A <u>theory</u> T is a set of sentences of first order logic (called axioms of the theory). Any sentence provable from axioms is called a <u>theorem of T</u>.

So far in this class, we have proved sentences in the "empty theory" ~ theory w/ no axioms.

(e.g. any time you proved

$\vdash S$)

First Order Arithmetic

A theory with some useful predicates:

<u>predicates</u>: $+$, $\cdot$, $S(x)$ S means "successor"

$a+b$ not $+(a,b)$

<u>name</u> : 0

We will <u>NOT</u> have "1", "2", "3"

Instead we have $\underbrace{S0, S(S0), S(SS0), \ldots}_{\text{names in theory}}$

Axioms of First Order Arithmetic

1. $\forall x \ \neg(0 = Sx)$
2. $\forall x \forall y \ (Sx = Sy \rightarrow x = y)$

   encodes "$x+1 = y+1 \rightarrow x = y$"
3. $\forall y \ (y = 0 \ \vee \ \exists x (Sx = y))$
4. $\forall x \ (x + 0 = x)$
5. $\forall x \forall y \ (x + Sy = S(x+y))$

   encodes $x+(y+1) = (x+y)+1$
6. $\forall x \ (x \cdot 0 = 0)$
7. $\forall x \forall y \ (x \cdot Sy = (x \cdot y) + x)$

   encodes $x(y+1) = xy + x$
8. $\forall x \forall y \ (x + y = y + x)$
9. $\forall x \forall y \ (x \cdot y = y \cdot x)$

standard interpretation

domain: $\{0, 1, 2, 3, \ldots\}$

$+$: addition

$\cdot$: multiplication

Let's prove FOA $\vdash$ SO+SO = SSO

1st order arithm

assume these premises & don't write them

1	$\forall y(S0 + Sy = S(S0+y))$	$\forall$E Ax 5 (x = S0)
2	$S0 + S0 = S(S0+0)$	$\forall$E 1 (y = 0)
3	$S0 + 0 = S0$	$\forall$E Ax 4 (x = S0)
4	$S0+S0 = SS0$	=E 3, 2

Prove

$$FOA \vdash SSO \cdot SSO = SSSSO$$

1	$\forall y(SSO \cdot Sy = (SSO \cdot y) + SSO)$	$\forall E$ Ax 7 $x = SSO$
2	$SSO \cdot SSO = (SSO \cdot SO) + SSO$	$\forall E$ 1 $y = SO$
3	$SSO \cdot SO = (SSO \cdot 0) + SSO$	$\forall E$ 1 $y = 0$
4	$SSO \cdot 0 = 0$	$\forall E$ Ax 6 $x = SSO$
5	$SSO \cdot SO = \boxed{0 + SSO}$	$=E$ 4, 3
6	$SSO + 0 = SSO$	$\forall E$ Ax 4 $x = SSO$
7	$\forall y(SSO + y = y + SSO)$	$\forall E$ Ax 8 $x = SSO$
8	$SSO + 0 = 0 + SSO$	$\forall E$ 7 $y = 0$
9	$0 + SSO = SSO$	$=E$ 8, 6
10	$SSO \cdot SO = SSO$	$=E$ 9, 5
11	$SSO \cdot SSO = SSO + SSO$	$=E$ 10, 2
12	$\forall y(SSO + Sy = S(SSO + y))$	$\forall E$ Ax 5 $x = SSO$
13	$SSO + SSO = S(SSO + SO)$	$\forall E$ 12 $y = SO$
14	$SSO + SO = S(SSO + 0)$	$\forall E$ 12 $y = 0$
15	$SSO + SO = SSSO$	$=E$ 6, 14
16	$SSO + SSO = SSSSO$	$=E$ 15, 13
17	$SSO \cdot SSO = SSSSO$	$=E$ 16, 11

5

$FOA \vdash \forall x(x \cdot (x+S0) = x \cdot x + x)$ $x(x+1) = x^2 + x$

1	$\forall y(a \cdot Sy = (a \cdot y) + a)$	$\forall E$ Ax 7	$x=a$
2	$a \cdot Sa = (a \cdot a) + a$	$\forall E$ 1	$y=a$
3	$\forall y(a + Sy = S(a+y))$	$\forall E$ Ax 5	$x=a$
4	$a + S0 = S(a+0)$	$\forall E$ 3	$y=0$
5	$a + 0 = a$	$\forall E$ Ax 4	$x=a$
6	$a + S0 = Sa$	$=E$ 5,4	
7	$a \cdot (a+S0) = a \cdot a + a$	$=E$ 6,2	
8	$\forall x(x \cdot (x+S0) = x \cdot x + x)$	$\forall I$ 7	

Sets

Mathematical notion of a collection of objects.

Sets are "unordered collections"

Have 2 fundamental properties:

1. two sets have same members iff they are the same set

2. given any predicate P, the set of x in universe such that $P(x)$ is true should be a set

$\pi \in \mathbb{R}$

Axioms of (naive) set theory

~ predicate $\in$ "element of"

1. Extensionality

$$\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x = y$$

2. (Unrestricted) Comprehension (Axiom schema)

For any predicate P:

$$\exists z \forall x (x \in z \leftrightarrow P(x))$$