Find first three nonzero terms of Taylor series for tan(x) centered at $c = \frac{\pi}{4}$ 1

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

Taylor theorem:

$$\tan(x) = \frac{\tan\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^0}{0!} + \frac{\tan'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^1}{1!} + \frac{\tan''\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^2}{2!} + \dots$$

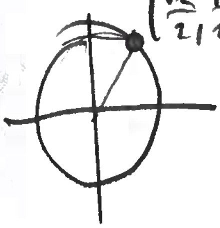
$0! = 1$

$$\tan'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{1}{\cos^2 x} = \cos^{-2}(x)$$

$$\Rightarrow \tan'\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2(\pi/4)} = \frac{1}{2/4} = \frac{1}{1/2} = 2$$

$$\tan''(x) = -2\cos^{-3}(x)(-\sin(x)) = \frac{2\sin(x)}{\cos^3(x)}$$

$$\Rightarrow \tan''\left(\frac{\pi}{4}\right) = \frac{2\sin(\pi/4)}{\cos^3(\pi/4)} = \frac{2(\cancel{\sqrt{2}/2})}{(\sqrt{2}/2)^{\cancel{3}\,2}} = \frac{2}{2/4} = 4$$



$\Rightarrow$ Taylor series of tan(x) centered at $\frac{\pi}{4}$ is...

$$\tan(x) = 1 + 2\left(x-\frac{\pi}{4}\right) + \frac{4\left(x-\frac{\pi}{4}\right)^2}{2} + \dots$$

$$= 1 + 2\left(x-\frac{\pi}{4}\right) + 2(x-\pi/4)^2 + \dots$$

Wolfram: $\tan'''(\pi/4) = 16$ $\quad \tan''''(\pi/4) = 80$ $\quad \tan^{(5)}(\pi/4) = 512$

Could improve approx:

$$\tan x = 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{80}{4!}\left(x-\frac{\pi}{4}\right)^4 + \frac{512}{5!}\left(x-\frac{\pi}{4}\right)^5 + \dots$$

3 nonzero terms of

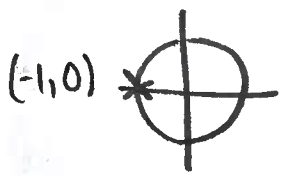
Taylor series of $f(x) = \sin(\pi x)\ln(x)$

Centered at $x=1$.

$$k=0 \rightarrow f^{(0)}(1) = \sin(\pi)\underbrace{\ln(1)}_{=0} = 0$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)(x-1)^k}{k!}$$

$$f'(x) = \pi\cos(\pi x)\ln(x) + \frac{\sin(\pi x)}{x}$$



$$= \frac{-2\pi(x-1)^2}{2!} + \frac{3\pi(x-1)^3}{3!} + \frac{4\pi(\pi^2-2)(x-1)^4}{4!}$$

$$k=1 \rightarrow f'(1) = \pi\cos(\pi)\underbrace{\ln(1)}_{=0} + \frac{\overbrace{\sin(\pi)}^{=0}}{1}$$

$f''(x) =$ complicated... use computer

$$-\frac{10\pi(\pi^2-3)(x-1)^5}{5!}$$

$$f''(1) = -2\pi$$

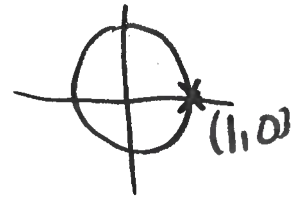
$$f^{(3)}(1) = 3\pi$$

$$f^{(4)}(1) = 4\pi(\pi^2-2)$$

$$f^{(5)}(1) = -10\pi(\pi^2-3)$$

Again $f(x) = \sin(\pi x)\ln(x)$

but center at $x=2$



$$f(2) = \underbrace{\sin(2\pi)}_{=0}\ln(2)$$
$$= 0$$

$$f'(2) = \pi\ln(2)$$

$$f''(2) = \pi$$

$$f'''(2) = -23.8481$$

$$f^{(4)}(2) = -58.871$$

$$f^{(5)}(2) = 283.742$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(2)(x-2)^2}{k!}$$

$$= \frac{\pi\ln(2)}{1!}(x-2)$$

$$+ \frac{\pi(x-2)^2}{2!}$$

$$- \frac{23.8481(x-2)^3}{3!}$$

$$- \frac{58.871(x-2)^4}{4!} + \frac{283.742(x-2)^5}{5!} + \ldots$$

Geometric series

centered at 0

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \,, \quad |r|<1$$

$$-1<r<1$$

Ex: Find power series for $f(x) = \frac{1}{3-x}$ centered at $c=1$

Notice: $f(x)$ looks sort of like $\frac{1}{1-x}$

$$f(x) = \frac{1}{3-x}\left(\frac{1/3}{1/3}\right)$$

$$= \frac{1}{3}\,\frac{1}{1-\left(\frac{x}{3}\right)}$$

geometric... but it would be

$$\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k$$

not centered at 1

fails

Instead:

$$f(x) = \frac{1}{3-x} = \frac{1}{3-(x+1-1)}$$

keep

$$= \frac{1}{2-(x-1)}$$

$$= \frac{1}{2-(x-1)}\left(\frac{1/2}{1/2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{1-\left(\frac{x-1}{2}\right)}$$

looks geometric!

geometric series w/ $r = \frac{x-1}{2}$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{x-1}{2}\right)^k = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}(x-1)^k$$

centered at 1