Taylor series ~ one way to generate a power series for a given function $f(x)$

Idea: Suppose we want to write

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \quad \text{centered at } c$$

Write out terms:

unknown constants

$$(\star) \quad f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \dots$$

chosen in advance

Goal: find unknown constants

Notice when we substitute $x=c$ into $(\star)$:

$$f(c) = a_0 + a_1\underbrace{(c-c)}_{=0} + a_2\underbrace{(c-c)^2}_{=0} + 0 + 0 + \dots$$

$$\boxed{a_0 = f(c)}$$

Take $\frac{d}{dx}(\star)$:

$$(\star\star) \quad f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \dots$$

Again substitute $x=c$ into $(\star\star)$:

$$f'(c) = a_1 + 0 + 0 + \dots$$

$$\Rightarrow \boxed{a_1 = f'(c)}$$

Take $\frac{d}{dx}(\star\star)$:

$$(\star\star\star) \qquad f''(x) = \underbrace{2}_{=2!}a_2 + 3!\,a_3(x-c) + 4\cdot 3(x-c)^2 + \cdots$$

Subst. $x=c$ into $(\star\star\star)$:

$$f''(c) = 2!\,a_2 + 0 + 0 + \cdots$$

$$\Rightarrow \boxed{a_2 = \frac{f''(c)}{2!}}$$

Take $\frac{d}{dx}(\star\star\star)$:

$$(\star\star\star\star) \qquad f'''(x) = 3!\,a_3 + 4!\,(x-c) + \cdots$$

Subst $x=c$ into $(\star\star\star\star)$:

$$f'''(c) = 3!\,a_3 + 0 + 0 + \cdots$$

$$\Rightarrow a_3 = \frac{f'''(c)}{3!}$$

note $f^{(0)}(c)$ means just $f(c)$

The pattern continues! In general,

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$$

n^{th} derivative of f (NOT a power) (keep parentheses!)

So we have Taylor series formula, centered at c:

Taylor theorem: $$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)\,(x-c)^k}{k!}$$

Earlier: we found Taylor series for e^x centered at 0:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Now: find Taylor series of e^x centered at $c=2$.

Here: $f(x)=e^x$, $c=2$

$$e^x = \sum_{k=0}^{\infty} \underbrace{a_k}_{\text{unknown}}(x-2)^k \underbrace{=}_{\text{Taylor Theorem}} \sum_{k=0}^{\infty} \frac{\overbrace{f^{(k)}(2)}^{\text{compute this}}(x-2)^k}{k!}$$

Notice:

$$\begin{aligned}
f^{(0)}(x)&=e^x \\
f'(x)&=e^x \to f'(2)=e^2 \\
f''(x)&=e^x \to f''(2)=e^2 \\
f'''(x)&=e^x \to f'''(2)=e^2 \\
&\vdots \qquad\qquad \vdots \\
f^{(k)}(x)&=e^x \quad f^{(k)}(2)=e^2
\end{aligned}$$

So we have

$$e^x = \sum_{k=0}^{\infty} \frac{e^2(x-2)^k}{k!} \rightsquigarrow$$

Ratio test shows this series has

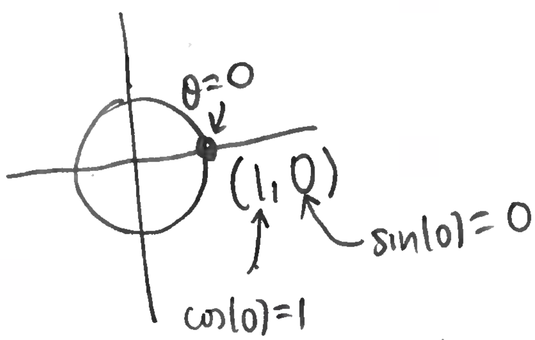
IoC: $\mathbb{R}$

RoC: ∞

Ex: Find Taylor series of $f(x) = \sin(x)$ centered at $c = 0$.

Soln: $\sin(x) = \sum_{k=0}^{\infty} a_k x^k \underset{\substack{\uparrow \\ \text{Taylor Thm}}}{=} \sum_{k=0}^{\infty} \frac{\sin^{(k)}(0)}{k!} x^k$

$$\begin{array}{ccc}
 & \underset{(k=0,4,8,\dots)}{\sin x} & \\
\overset{\frac{d}{dx}}{\nearrow} & & \searrow^{\frac{d}{dx}} \\
\underset{(k=3,7,11,\dots)}{-\cos x} & & \underset{(k=1,5,9,\dots)}{\cos x} \\
\uparrow^{\frac{d}{dx}} & & \swarrow_{\frac{d}{dx}} \\
 & \underset{(k=2,6,10,\dots)}{-\sin(x)} &
\end{array}$$



k	$\sin^{(k)}(0)$
0	0
1	1
2	0
3	-1
4	0
5	1
6	0
7	-1
8	0

→ this repeats

Therefore,

$$\sin(x) = 0 + \frac{1}{1!}x + 0 + \frac{(-1)}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 + \frac{(-1)}{7!}x^7 + \dots$$
$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots$$