Ex: Find IoC and RoC of

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(k+1)(k+2)}$$

Soln: Ratio test: $a_k = \dfrac{(-1)^k x^k}{(k+1)(k+2)}$

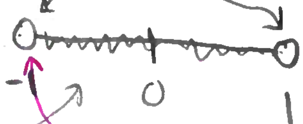
Compute

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{\cancel{(-1)^{k+1}}\, \cancel{x^{k+1}}}{\cancel{(k+2)}(k+3)} \cdot \frac{(k+1)\cancel{(k+2)}}{\cancel{(-1)^k}\, \cancel{x^k}} \right|$$

$$= \lim_{k\to\infty} |x| \left(\frac{k+1}{k+3}\right) \quad \frac{\infty}{\infty}$$

$$\overset{L'H}{=} |x| \lim_{k\to\infty} \frac{1}{1} = |x| < 1$$

enforce this condition to get convergence

$\Rightarrow$ 

so far, endpoint behavior is not known

x-values

Check endpoints:

$x = -1$

Plug into series:

$$\sum_{k=0}^{\infty} \frac{(-1)^k \cancel{(-1)^k}}{(k+1)(k+2)} = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}$$

" $\tilde{a}_k$ "

Limit comparison test: $b_k = \dfrac{1}{k^2}$

Compute

$$\lim_{k\to\infty} \frac{\tilde{a}_k}{b_k} = \lim_{k\to\infty} \left(\frac{1}{(k+1)(k+2)} \cdot \frac{k^2}{1}\right)$$

$$= \lim_{k\to\infty} \frac{k^2}{k^2+3k+2} = 1 \Rightarrow \text{LCT, } \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} \text{ converges}$$

$\Rightarrow$ $\boxed{\text{power series converges at } x = -1}$

<u>other endpt at $x=1$</u>

$$\sum_{k=0}^{\infty} \frac{(-1)^k 1^k}{(k+1)(k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)}$$

To use alt. series test; compute

$$\lim_{k\to\infty} \frac{1}{(k+1)(k+2)} = 0$$

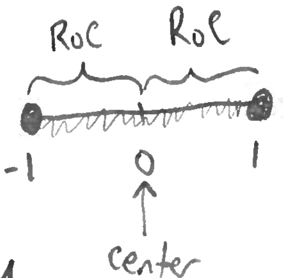
And so series converges by alt. series test.

$\Rightarrow$ power series converges when $x=1$

<u>Conclusions</u> The power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(k+1)(k+2)}$

has

IoC:



(or $[-1,1]$ or $-1 \le x \le 1$)

RoC: $\underline{1}$

$2(k+1)$

Ex: $\displaystyle\sum_{k=0}^{\infty} (2k)!\left(\frac{x}{3}\right)^k$

$(2k+2)! = (2k+2)(2k+1)\boxed{(2k)(2k-1)\cdots 2\cdot 1}$

$\boxed{(2k)(2k-1)\cdots 2\cdot 1} = (2k)!$

$(2k+2)! = (2k+2)(2k+1)(2k)!$

Ratio test: $a_k = (2k)!\left(\frac{x}{3}\right)^k$

Compute

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(2k+2)!\,x^{k+1}}{3^{k+1}}\,\frac{3^k}{(2k)!\,x^k}\right|$$

$$= \lim_{k\to\infty}\frac{(2k+2)(2k+1)\,|x|}{3}$$

$$= \infty \Rightarrow \text{diverges!}$$

IoC: $\{0\}$

RoC: 0

$(k+1)! = (k+1)k!$

Ex: Bessel function of order 0 is

$$J_0(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$$

Ratio test: $a_k = \dfrac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \Rightarrow$ compute

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}\,x^{2k+2}\,\cancel{2^{2k}}\,(k!)^2}{2^{2(k+1)}\left((k+1)!\right)^2\,\cancel{(-1)^k}\,x^{2k}}\right|$$

$$= \lim_{k\to\infty}\frac{|x|^2}{(k+1)^2\,4} = 0 < 1$$

$\Rightarrow$ converge no matter what value that x takes!!

$$\Rightarrow \boxed{\begin{array}{l}\text{IoC: } \mathbb{R}\\ \text{RoC: } \infty\end{array}} \quad \left(\begin{array}{l}\text{or } (-\infty,\infty)\\ \text{or } \longleftrightarrow\\ \text{or } -\infty < x < \infty\end{array}\right)$$

Student Research Project

this page differs from lecture because I found a mistake!! It's now correct

$$Li_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

Ratio test

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)^2} \frac{k^2}{\cancel{x^k}} \right|$$

$$= \lim_{k\to\infty} |x| \frac{k^2}{k^2+2k+1} = |x| \overset{\text{enforce}}{<} 1$$

RoC: 1

IoC: $(-1, 1)$

$$x^{\underline{2}} = x(x-1)$$

<u>Define</u>: falling powers

$$x^{\underline{n}} = \underbrace{x(x-1)(x-2)\dots(x-n+1)}_{n \text{ factors}}$$

Define "discrete Li_2"

$$\rightarrow li_2(x) = \sum_{k=1}^{\infty} \frac{x^{\underline{k}}}{k^2}$$

<u>Ratio test</u>: $\displaystyle \lim_{k\to\infty} \frac{|x|^{\underline{k+1}}}{(k+1)^2} \frac{k^2}{|x|^{\underline{k}}}$

$$= \lim_{k\to\infty} \frac{\cancel{|x|}\,\cancel{|x-1|}\dots\cancel{|x-k+1|}\,|x-k|}{\cancel{|x|}\,\cancel{|x-1|}\dots\cancel{|x-k+1|}} \underbrace{\frac{k^2}{(k+1)^2}}_{\to 1}$$

$$= \lim_{k\to\infty} \underbrace{|x-k|}_{\to\infty} \underbrace{\frac{k^2}{(k+1)^2}}_{\to 1} = \infty \Rightarrow \text{divergent!!}$$

(note: does not diverge at $x = 1, 2, 3, \dots$, it turns out, because the sum becomes finite there!)