Most general form of power series: variable terms look like $x^{\text{power dep on } k)}$

x is variable

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \leftarrow \text{power series "centered at 0"}$$

OR

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \leftarrow \text{power series "centered at c"}$$

Ex: (exponential)

What if we wanted a power series for e^x

"centered at 0"

We need to enforce

$$e^x = \sum_{k=0}^{\infty} a_k x^k$$

in other words

unknowns

$$(\star) \longrightarrow e^x = \underline{a_0} + \underline{a_1} x + \underline{a_2} x^2 + \underline{a_3} x^3 + \dots$$

To find a_0: take $x=0$

$$1 \underset{\uparrow}{=} e^0 \overset{(\star)}{=} a_0 + \underbrace{0a_1 + 0a_2 + \dots}_{\text{all } = 0} \Rightarrow \boxed{1 = a_0}$$

compute

So now we have

unknowns

$$(\star\star) \quad e^x = 1 + \underline{a_1} x + \underline{a_2} x^2 + \underline{a_3} x^3 + \dots$$

From (★★), take $\frac{d}{dx}$:

$$\frac{d}{dx}e^x = \frac{d}{dx}\left[\overset{0}{1} + a_1\overset{1}{x} + a_2\overset{2x}{x^2} + a_3\overset{3x^2}{x^3} + a_4x^4 + \cdots\right]$$

(★★★) $$e^x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

Plug in $x=0$ into (★★★):

$$e^0 = a_1 + \underbrace{0a_2 + 0a_3 + \cdots}_{\text{all zero}} \longrightarrow \boxed{1 = a_1}$$

So now we update (★★) and see

(★★★★) $$e^x = 1 + x + a_2x^2 + a_3x^3 + \cdots$$

So compute $\frac{d}{dx}$((★★★)) to get

$$e^x = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \cdots$$

Plug in $x=0$:

$$e^0 = 2a_2 + \underbrace{0 + 0 + 0 + \cdots}_{\text{all zero}} \longrightarrow 1 = 2a_2$$

$$\rightarrow \boxed{a_2 = 1/2} = \frac{1}{2!}$$

Another $\frac{d}{dx}$:

$$e^x = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4x + \cdots$$

$$\Big\downarrow x=0$$

$$1 = 3\cdot 2a_3 \longrightarrow a_3 = \frac{1}{3\cdot 2} = \frac{1}{3!}$$

Another $\frac{d}{dx}$: $e^x = 4!a_4 + \cdots \xrightarrow{x=0} 1 = 4!a_4 \rightarrow a_4 = \frac{1}{4!}$

Pattern is clear: note: $0! = 1$ by definition

$$a_0 = 1 = \frac{1}{0!}$$

$$a_1 = 1 = \frac{1}{1!}$$

$$a_2 = \frac{1}{2} = \frac{1}{2!}$$

$$a_3 = \frac{1}{3!}, \; a_4 = \frac{1}{4!} \ldots \text{ so } a_k = \frac{1}{k!}$$

So update (★★★★) to get

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

So, for $x=2$:

$$e^2 = \sum_{k=0}^{\infty} \frac{2^k}{k!}$$ ~ does that series converge?

Ratio test: $a_k = \frac{2^k}{k!}$

$(k+1)! = (k+1)(k!)$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left(\frac{\cancel{2^{k+1}}}{\cancel{(k+1)!}} \cdot \frac{\cancel{k!}}{\cancel{2^k}}\right)$$

$$= \lim_{k\to\infty} \frac{2}{\cancel{k}+1} = 0 < 1$$

→ series conv. abs.

Earlier: we had geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$ ~ ONLY VALID when $-1 < x < 1$

Now: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ ~ where is it valid? i.e. which x can we plug into the series?

Apply ratio test to $\sum_{k=0}^{\infty} \frac{x^k}{k!}$: $a_k = \frac{x^k}{k!}$

and compute

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left(\frac{\cancel{x^{k}}\,x}{(k+1)!}\right)\left(\frac{k!}{\cancel{x^k}}\right)$$

$$= \lim_{k\to\infty} \frac{x}{k+1} = 0 < 1$$

$\Rightarrow$ series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for <u>ANY</u> $x \in \mathbb{R}$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots$$

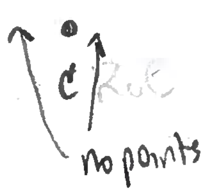
<u>Def</u> : Any power series $\sum_{k=0}^{\infty} a_k (x-\overset{\text{center}}{c})^k$ has an associated set of x-values for which it converges. ALWAYS: $x=c$ ~ the "center" is in the set.

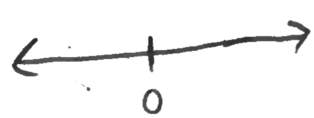
The whole set of x for which series converges is called region of convergence.

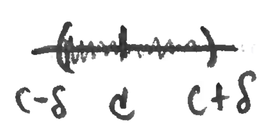
Turns out: regions of convergence ALWAYS look like

$\{c\}$ $\qquad (-\infty, \infty)$ OR $(c-\delta, c+\delta)$

$\delta > 0$ is a fixed value







Ex: Consider power series $\sum_{k=0}^{\infty} k!\, x^k$

Find region of conv.

① centered at 0: will converge at $x=0$

② other points? (let $x \neq 0$)

Ratio test: $a_k \equiv k!\, x^k$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!\, x^{\cancel{k+1}}}{k!\, \cancel{x^k}}\right|$$

$$= \lim_{k\to\infty}\left|\underbrace{(k+1)}_{\infty}\ \underbrace{x}_{\text{const.} \neq 0}\right| = \infty > 1$$

So ratio test concludes the series diverges for all $x \neq 0$.

Three good examples

Power Series	Region of Convergence	
$\left(e^x =\right) \sum_{k=0}^{\infty} \frac{x^k}{k!}$	$\mathbb{R}$; 0 ; $-\infty < x < \infty$	$[-1,1]$ includes $1,-1$
$\left(\frac{1}{1-x} =\right) \sum_{k=0}^{\infty} x^k$	$(-1,1)$; -1 0 1 ; $-1<x<1$	$]-1,1[\; = (-1,1)$ not include $-1,1$
$\sum_{k=0}^{\infty} k!\, x^k$	$\{0\}$; 0 ; $0 \le x \le 0$	