Conv or div? (using root test)

Ex: $\sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k \qquad a_k = \left(\frac{1}{5}\right)^k$

Soln: Use root test:

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \cancel{\sqrt[k]{\left(\tfrac{1}{5}\right)^{k}}}$$

$$= \lim_{k\to\infty} \frac{1}{5} = \frac{1}{5} < 1$$

Therefore the series $\sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k$ conv. absolutely.

Ex: $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$

Soln: Use root test:

$$\lim_{k\to\infty} \cancel{\sqrt[k]{\left(1+\tfrac{1}{k}\right)^k}} = \lim_{k\to\infty} 1 + \frac{1}{k} = 1$$

The root test is inconclusive!

A different test needed!

Ex: $\sum_{k=1}^{\infty} \left(\frac{3k+2}{k+3}\right)^k$

Soln: Use root test: $\lim_{k\to\infty} \sqrt[k]{\left|\left(\frac{3k+2}{k+3}\right)^k\right|}$

$$= \lim_{k\to\infty} \frac{3k+2}{k+3} \overset{L'H}{=} 3 > 1$$

Therefore series $\sum_{k=1}^{\infty} \left(\frac{3k+2}{k+3}\right)^k$ diverges!

Ex: $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{(\ln(k))^k}$

Could use alt. series test

$(-1)^k \sim -1, 1, -1, 1$

Using ~~ratio~~ root test,

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k}{(\ln(k))^k}\right|}$$

$$= \lim_{k\to\infty} \cancel{\sqrt[k]{}}\!\!\sqrt{\frac{1}{\ln(k)^{\cancel{k}}}}$$

$$= \lim_{k\to\infty} \frac{1}{\ln(k)} = 0 < 1 \Rightarrow$$ Therefore the series converges.

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^k}{(k^k)^2}$

note: $(k^k)^2 = k^{2k} = (k^2)^k$

$$\Rightarrow \boxed{\frac{(k!)^k}{(k^k)^2} = \left(\frac{k!}{k^2}\right)^k}$$

$k! = k(k-1)(k-2)\dots(2)(1)$

$(k-(k-2))(k-(k-1))$

$\approx k^k + \dots$

$\overset{\text{root test}}{\Longrightarrow} \displaystyle\lim_{k\to\infty} \sqrt[k]{\frac{(k!)^k}{(k^k)^2}}$

$$= \lim_{k\to\infty} \cancel{\sqrt[k]{}}\!\!\sqrt{\left(\frac{k!}{k^2}\right)^{\cancel{k}}}$$

$$= \lim_{k\to\infty} \frac{k!}{k^2} = \infty$$

$\Rightarrow$ Therefore $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^k}{(k^k)^2}$ diverges

Power series

<u>Question</u>: How can I use series to express functions?

<u>Recall</u>: <u>geometric series</u>

if $|r|<1$, then $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

Think about letting r be a variable — say, x.

$$\underbrace{\frac{1}{1-x}}_{f(x)} = \underbrace{\boxed{\sum_{k=0}^{\infty} x^k}}_{\text{series expression for } f(x)} = 1+x+x^2+x^3+x^4+\cdots$$

Only valid when $|x|<1$

$-1<x<1$