Ratio test

Let $\sum^{\infty} a_k$ be a series with nonzero terms.

1. $\sum^{\infty} a_k$ converges absolutely when $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$
2. $\sum^{\infty} a_k$ diverges when $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$
3. test inconclusive if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$

Factorials

$3! = 3\cdot2\cdot1 = 6$

$4! = 4\cdot3\cdot2\cdot1 = 24$

$5! = 5\cdot4\cdot3\cdot2\cdot1 =$

$n! = n(n-1)\ldots 2\cdot1$

Ex: $\sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k$ ~ $a_k = \left(\frac{1}{5}\right)^k$

geometric $r = \frac{1}{5}$ $\Rightarrow$ converge

So compute

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(1/5)^{k+1}}{(1/5)^k}$$

$$= \lim_{k\to\infty}\frac{5^k}{5^{k+1}}$$

$$= \lim_{k\to\infty}\frac{1}{5} = \frac{1}{5} < 1$$

So by ratio test, the series converges.

Ex: $\sum_{k=0}^{\infty} \frac{2^k}{k!}$ ~ here $a_k = \frac{2^k}{k!}$

So compute $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\left(\frac{2^{k+1}}{(k+1)!}\right)}{\left(\frac{2^k}{k!}\right)}$

$(k+1)! = (k+1)\underbrace{k(k-1)\ldots 2\cdot1}_{=k!}$

$(k+1)! = (k+1)k!$

$$= \lim_{k\to\infty}\left(\frac{2^{k+1}}{(k+1)!}\right)\left(\frac{k!}{2^k}\right)$$

$$= \lim_{k\to\infty}\frac{2}{k+1} = 0 < 1$$

So by ratio test, $\sum_{k=0}^{\infty}\frac{2^k}{k!}$ converges.

(in fact, it conv. to e^2)

$2^x \ll 3^x \ll \ldots \ll k!$

$2^x \ll 3^x \ll \ldots \ll x^x$

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ ~ here, $a_k = \dfrac{k!}{k^k}$

"Superexponential"
variable base,
variable power

So compute

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\overset{(k+1)k!}{(k+1)!}}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} \right|$$

x^2+x

$$(k+1)^k = \overbrace{(k+1)(k+1)(k+1)\ldots(k+1)}^{k \text{ times}}$$
$$= k^k + \text{lower order terms}$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{(k+1)}\, k^k}{(k+1)^{k+\cancel{1}}} \right|$$

$$= \lim_{k\to\infty} \frac{k^k}{(k+1)^k}$$

$$= \lim_{k\to\infty} \frac{k^k}{k^k + \underbrace{\text{lower order terms}}_{\to\infty \text{ as } k\to\infty}}$$

$= \cancel{1?}$

$\boxed{\text{PLOT}} = 0 < 1$

→ By ratio test, the series converges.

$\log(x)$

$x \ll x^2 \ll x^3 \ll \cdots \ll 2^x \ll e^x \ll 3^x \ll \ldots \ll k! \ll k^k$

Ex: $\sum_{k=1}^{\infty} \frac{5^k}{k^4}$ ~ here $a_k = \frac{5^k}{k^4}$

So compute

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{5^{k+1}}{(k+1)^4} \cdot \frac{k^4}{5^k} \right|$$

$$= \lim_{k\to\infty} \left| 5 \underbrace{\left(\frac{k}{k+1}\right)^4}_{\to 1} \right|$$

$$= 5 > 1$$

So by ratio test, series diverges.

Ex: $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k}}_{\substack{\text{harmonic} \\ \text{series} \\ \text{(diverges)}}}$ ~ here $a_k = \frac{1}{k}$

So compute

$$\lim_{k\to\infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

So ratio test $\to$ inconclusive.

Ex: $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\substack{p\text{-series} \\ p=2 \to \text{conv.}}}$ ~ here $a_k = \frac{1}{k^2}$

So compute

$$\lim_{k\to\infty} \left| \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} \right| = \lim_{k\to\infty} \frac{k^2}{k^2+2k+1} = 1$$

So ratio test $\to$ inconclusive!

Useful limit: $\lim_{k\to\infty} \sqrt[k]{k} = \lim_{k\to\infty} k^{1/k} = 1$

$1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots$

How is this true?

$$k^{1/k} = e^{\ln\left(k^{1/k}\right)} = e^{\frac{1}{k}\ln(k)}$$

Observe: $\lim_{k\to\infty} \frac{1}{k}\ln(k) \overset{LH}{=} \lim_{k\to\infty} \frac{\frac{1}{k}}{1} = 0$

Therefore: $\lim_{k\to\infty} \sqrt[k]{k} = \lim_{k\to\infty} e^{\frac{\ln(k)}{k}} = e^0 = 1$

$\frac{\infty}{\infty}$

equal

Theorem (Root test)

1. $\sum a_k$ converges absolutely if $\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$
2. $\sum a_k$ diverges if $\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1$ or $(=\infty)$
3. inconclusive if $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$