Ex: Conv or diverge?

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}} \leftarrow a_k$$

Compare with $b_k = \frac{1}{k^2}$

Compute

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k\sqrt{k^2+1}}}{\frac{1}{k^2}}$$

$$= \lim_{k\to\infty} \frac{k^2}{k\sqrt{k^2+1}} (1)$$

$$k = \sqrt{k^2}$$

$$= \lim_{k\to\infty} \left(\frac{\cancel{k^2}}{\cancel{k}\sqrt{k^2+1}}\right)\left(\frac{\frac{1}{\cancel{k^2}}}{\frac{1}{\cancel{k}}}\right)$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{\frac{k^2+1}{k^2}}}$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{1+1/k^2}} \qquad 1/k^2 \to 0$$

$$= 1 \neq 0$$

$\Rightarrow$ by LCT $\sum^{\infty} \frac{1}{k\sqrt{k^2+1}}$ does same as $\sum^{\infty} \frac{1}{k^2}$

But $\sum^{\infty} \frac{1}{k^2}$ converges b/c it is a p-series w/ $p>1$

Therefore, $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ converges.

Conv or div?

$$\sum_{k=1}^{\infty} \frac{k}{k^2+1} \quad \swarrow a_k$$

Pick $b_k = \frac{1}{k}$.

Compute

$$\lim_{k\to\infty} \frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2}{k^2+1} = 1 \neq 0$$

Therefore, LCT tells us that $\sum \frac{k}{k^2+1}$ does same as $\sum \frac{1}{k}$

BUT $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic series)

So, $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ also diverges

<u>Alternating series test</u>

exception: test for divergence

Recall: all tests so far required terms of series to be positive

<u>Def</u>: A series $\sum^{\infty} a_k$ is called <u>alternating</u> if

$a_k = (-1)^k b_k$ or $a_k = (-1)^{k+1} b_k$ for some positive series b_k.

(in short ~ + − + − + − or − + − + − +)

<u>Theorem</u> (Alternating Series Test)

Consider an alternating series $\sum (-1)^k b_k$ or $\sum (-1)^{k+1} b_k$

Then the series converges when

$$\lim_{k\to\infty} b_k = 0.$$

<u>Ex</u>: (Alternating harmonic series) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

Here: $b_k = \frac{1}{k}$. So compute $\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{1}{k} = 0$

So by AST, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

Conv or div?

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{3k+2}$$

Here, $b_k = \frac{1}{3k+2}$ and $\lim_{k\to\infty} \frac{1}{3k+2} = 0$,

so by AST, $\sum_{k=1}^{\infty} \frac{(-1)^k}{3k+2}$ converges

Conv or div?

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{3k+2}$$

Here, $b_k = \frac{k}{3k+2}$ and $\lim_{k\to\infty} b_k = \frac{1}{3} \neq 0$

$\Rightarrow$ series diverges

(think: this series gets close to something like $\frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots$ for large k)

Conv or div?

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k}{\ln(k+1)}$$

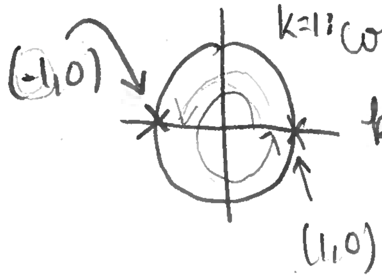
Here, $b_k = \frac{k}{\ln(k+1)}$ and so $\lim_{k\to\infty} b_k \overset{L'H}{\underset{\infty/\infty}{=}} \lim_{k\to\infty} \frac{1}{\frac{1}{k+1}}$

$= \lim_{k\to\infty} k+1 = \infty$

$\Rightarrow$ series diverges

Conv or div?

$$\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$ ← Converges as shown earlier



$k=1$: $\cos(\pi) = -1$

$k=2$: $\cos(2\pi) = 1$

$k=3$: $\cos(3\pi) = -1$

$\vdots$

Def: A series $\sum a_k$ is called conditionally convergent if $\sum a_k$ converges while $\sum |a_k|$ diverges.

If $\sum |a_k|$ converges, then we say that $\sum a_k$ is absolutely convergent.

Theorem: If $\sum a_k$ converges absolutely, then $\sum a_k$ converges.

Def: By a rearrangement of a series, we mean a series with same terms, but in a different order.

Theorem: (Riemann rearrangement theorem)

Suppose $\sum^{\infty} a_k$ is conditionally convergent.

Let M be any real number you want.

Then there is a rearrangement of $\sum^{\infty} a_k$, say $\sum^{\infty} b_k$, such that

$$M = \sum^{\infty} b_k$$

See 12 March notes p.3-4