Ex: Conv or div?

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$$

$\uparrow$

a_k

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

Pick $b_k = \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$

Compute

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{1}{\sqrt{k}+1}}{\frac{1}{\sqrt{k}}}$$

$$= \lim_{k \to \infty} \frac{\sqrt{k}}{\sqrt{k}+1} (\cdot 1)$$

$$= \lim_{k \to \infty} \frac{\sqrt{k}}{\sqrt{k}+1} \frac{(1/\sqrt{k})}{(1/\sqrt{k})}$$

$$= \lim_{k \to \infty} \frac{1}{1 + 1/\sqrt{k}} \quad 1/\sqrt{k} \to 0$$

$$= 1 \neq 0$$

$$\neq \infty$$

So by LCT, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ and $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ do same

But $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges since it is a p-series w/ $p < 1$.

Thus $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ also diverges.

Ex: Conv or div?

$$\sum_{k=1}^{\infty} \frac{2^k+1}{3^k} \leftarrow a_k$$

$$\sum_{k=1}^{\infty} f_k(n)$$

Pick $b_k = \left(\frac{2}{3}\right)^k = \frac{2^k}{3^k}$

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\downarrow$$

$$\frac{2^k+1}{2^k} = 1 + \frac{1}{2^k}$$

Compute

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{2^k+1}{3^k}}{\frac{2^k}{3^k}}$$

$$= \lim_{k\to\infty} \frac{2^k+1}{\cancel{3^k}} \cdot \frac{\cancel{3^k}}{2^k}$$

$$= \lim_{k\to\infty} 1 + \frac{1}{2^k} \to 0$$

$$= 1 \neq 0$$

By LCT, $\sum_{k=1}^{\infty} \frac{2^k+1}{3^k}$ does same as $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$

$$\frac{k+1}{k^2} = \frac{1}{k} + \frac{1}{k^2}$$

But $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ converges because it is geometric series with $|r| = \left|\frac{2}{3}\right| < 1$.

Therefore, $\sum_{k=1}^{\infty} \frac{2^k+1}{3^k}$ converges.

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2} \leftarrow a_k$

we know $\ln(x) \ll x \ll x^2$

$\ln|x| \ll x^p$

Pick $b_k = \frac{1}{k^2}$.

Compute

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln(k)}{k^2}}{\frac{1}{k^2}}$$

$$= \lim_{k\to\infty} \ln(k) = \infty$$

suggests (iii) of thm is useful

So $b_k = \frac{1}{k^2}$ fails to be useful!! BUT $\sum b_k = \sum \frac{1}{k^2}$ converges, contrary to what we need in thm

So pick $b_k = \frac{1}{k}$. Compute

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln(k)}{k^2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\ln(k)}{k} \quad \frac{\infty}{\infty}$$

$$\stackrel{L'H}{=} \lim_{k\to\infty} \frac{1/k}{1} = 0$$

suggests (ii) of the theorem

So $b_k = \frac{1}{k}$ fails to be useful again!! BUT $\sum b_k = \sum \frac{1}{k}$ diverges (harmonic)

So pick compromise b/w above attempts: $b_k = 1/k^{3/2}$. Compute

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln(k)}{k^2}}{1/k^{3/2}} = \lim_{k\to\infty} \frac{\ln(k)}{k^{1/2}} \stackrel{L'H}{=} \lim_{k\to\infty} \frac{1/k}{1/2k^{1/2}} = 2\lim_{k\to\infty} \frac{1}{k^{1/2}} = 0$$

So we are in (ii) in thm AND $\displaystyle\sum^{\infty} b_k = \sum^{\infty} \frac{1}{k^{3/2}}$ converges since it is a p-series. $p>1$.

Therefore by LCT $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ converges!

Ex: Conv or div?

$$\sum_{k=1}^{\infty} \underbrace{\ln\left(1+\frac{1}{k^2}\right)}_{a_k}$$

recall $\ln(a)+\ln(b)=\ln(ab)$

So this sum resembles the infinite product

$$\ln\left(\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)\right)$$

"Infinite product"

Pick $b_k=\frac{1}{k^2}$.

Compute $\displaystyle\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{\ln\left(1+\frac{1}{k^2}\right)}{1/k^2}$ $\quad \frac{0}{0}$

$$\overset{L'H}{=}\lim_{k\to\infty}\frac{\left(\frac{1}{1+1/k^2}\right)\cancel{\left(\frac{-2}{k^3}\right)}}{\cancel{\left(-2/k^3\right)}}$$

$$=\underline{1}\neq 0$$

So by LCT, $\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2}\right)$ does same as $\sum_{k=1}^{\infty}\frac{1}{k^2}$

BUT $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges since it's a p-series w/ $p>1$

Therefore, $\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2}\right)$ converges.