Examples of comparison test

Conv or div?

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+3k+1}$

$\sum \frac{1}{k^p}$ ~ conv when $p>1$, div when $p\leq 1$

FACT: $k^3+3k+1 > k^3$

$\Downarrow$

$\displaystyle\frac{1}{k^3+3k+1} < \frac{1}{k^3}$ p series w/ $p>1$ $\Rightarrow$ conv.

$\Downarrow$

$$0 < \sum_{k=1}^{\infty} \frac{1}{k^3+3k+1} < \sum_{k=1}^{\infty} \frac{1}{k^3}$$

$\Downarrow$

Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ conv, the comp. test shows that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+3k+1}$ also conv.

Ex: Converge or diverge?

$$\sum_{k=1}^{\infty} \frac{1}{2^k+1}$$

$$2^k+1 > 2^k$$

$$\downarrow$$

$$\frac{1}{2^k+1} < \frac{1}{2^k}$$

$$\left(\frac{1}{2}\right)^k$$

$$\downarrow$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k+1} < \underbrace{\sum_{k=1}^{\infty} \frac{1}{2^k}}$$

this converges b/c it is geometric with $r = 1/2$

$$\Downarrow$$

by comparison test, $\sum_{k=1}^{\infty} \frac{1}{2^k+1}$ conv.

Geometric

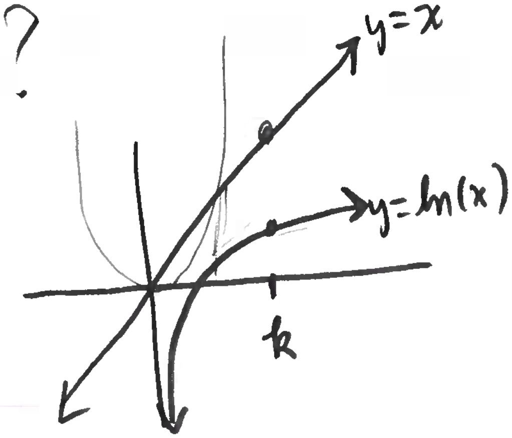
$\sum_{k=}^{\infty} r^k$ ~ conv when $|r|<1$

p-series

$$\sum^{\infty} \frac{1}{k^p}$$

Ex: Converge or diverge?

$$\sum_{k=2}^{\infty} \frac{1}{\ln(k)}$$



Notice from graph:

$$\ln(k) < k$$

$$\downarrow$$

$$\frac{1}{\ln(k)} > \frac{1}{k}$$

(essentially the) harmonic series

}

diverges

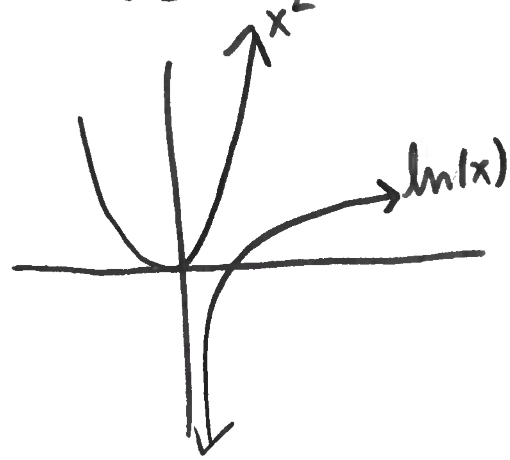
$$\downarrow$$

$$\sum_{k=2}^{\infty} \frac{1}{\ln(k)} > \sum_{k=2}^{\infty} \frac{1}{k}$$

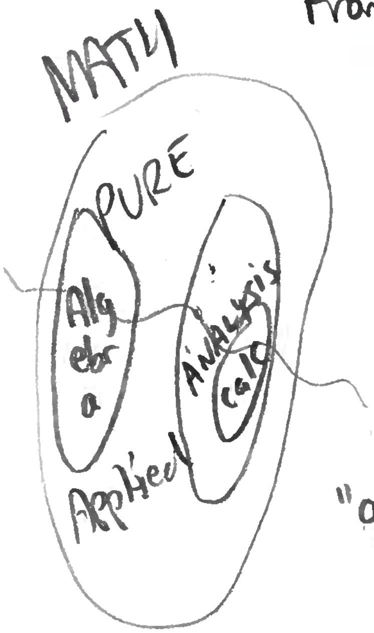
$$\Downarrow$$

Since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, conclude here that $\sum_{k=2}^{\infty} \frac{1}{\ln(k)}$ also diverges

Ex: Same as before, $\sum_{k=2}^{\infty} \frac{1}{\ln(k)}$, notice



From graph:



$$\ln(k) < k^2$$

$$\downarrow$$

$$\underset{\text{"}a_k\text{"}}{\frac{1}{\ln(k)}} > \underset{\text{"}b_k\text{"}}{\frac{1}{k^2}}$$

$$\downarrow$$

$$\underbrace{\sum_{k=2}^{\infty} \frac{1}{\ln(k)}}_{\text{already know FROM previous page this diverges}} > \sum_{k=2}^{\infty} \frac{1}{k^2}$$

converges ~ essentially p-series w/ $p > 1$

"analysis"

Hence in (ii) of thm in 26 March notes

Thm does not apply here: here $a_k \geq b_k$ but we don't know that $\sum b_k$ diverges (in fact, it converges)

So we cannot apply thm here!

From this $\rightsquigarrow$ NO CONCLUSION!!

Ex: Conv or diverge?

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1}$$

Soln:

$$k^2-1 < k^2$$

$$\downarrow$$

$$\frac{1}{k^2-1} > \frac{1}{k^2}$$

conv... p-series!

$$\downarrow$$

$$\underline{\underline{\sum_{k=2}^{\infty} \frac{1}{k^2-1}}} > \sum_{k=2}^{\infty} \frac{1}{k^2}$$

$$\downarrow$$

Cannot draw conclusion in this case!

Limit comparison test

More powerful cousin to the comparison test.

non-negative seqs

both conv or both div

Theorem: Let $a_k, b_k \geq 0$

i) if $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = L \neq 0$, then $\sum^{\infty} a_k$ and $\sum^{\infty} b_k$ do same

ii) if $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum^{\infty} b_k$ conv, then $\sum^{\infty} a_k$ conv

iii) if $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = \infty$ and $\sum^{\infty} b_k$ div, then $\sum^{\infty} a_k$ div

From 26 March:

$$\sum_{k=1}^{\infty} \frac{1}{k+1/2}$$ ~ comparison test failed to be applicable

Let $a_k = \frac{1}{k+1/2}$ and $b_k = \frac{1}{k}$ ← YOU choose b_k

Compute

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k+1/2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1/2} \quad \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{k\to\infty} \frac{1}{1} = \overset{L}{1} \neq 0$$

By Limit Comp Test, $\sum^{\infty} \frac{1}{k+1/2}$ does same as $\sum^{\infty} \frac{1}{k}$.

Since $\sum^{\infty} \frac{1}{k}$ is harmonic series, it diverges.

Therefore, $\sum^{\infty} \frac{1}{k+1/2}$ diverges.

Example (p.5 of these notes)

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1}$$ ~ couldn't conclude anything w/ comp. test

Let $a_k = \frac{1}{k^2-1}$ and $b_k = \frac{1}{k^2}$.

Compute

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1$$

LCT (ii)

Therefore, $\sum^{\infty} \frac{1}{k^2-1}$ does same as $\sum^{\infty} \frac{1}{k^2}$.

But, $\sum^{\infty} \frac{1}{k^2}$ converges since it's a p-series.

Therefore, $\sum_{k=2}^{\infty} k^2-1$ converges as well.