Note: in previous example from 24 March, we found $N = 22595$ by plotting

$$\frac{\ln(x)}{x} + \frac{1}{x}$$ in desmos

and $y = 0.001$

and finding where the curve goes below height 0.001.

By WolframAlpha we can compute

$$S_{22595} \overset{\text{def}}{=} \sum_{k=2}^{22595} \frac{\ln(k)}{k^2} = 0.9370603$$

should be accurate to about 3 dec places

"wrong"

computer

correct

actual ∞ sum is ≈ 0.93755

our estimate is within 0.001 of the "actual sum"

Ex: Estimate $\sum_{K=1}^{\infty} \frac{1}{1+K^2}$ ~~up to~~ with error $\epsilon = 0.0001$

Soln: By theorem (from 24 March notes) we have

$$\text{error} = R_N < \int_N^{\infty} \frac{1}{1+x^2}\,dx$$

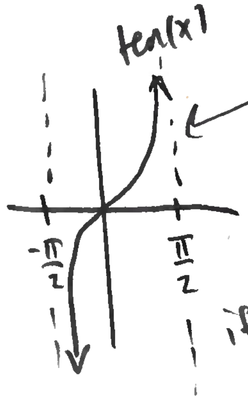
$$= \lim_{b\to\infty} \int_N^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \arctan(x)\Big|_N^b$$

$$= \lim_{b\to\infty} \left(\arctan(b) - \arctan(N)\right)$$

$$= \frac{\pi}{2} - \arctan(N)$$

we control N



increasing

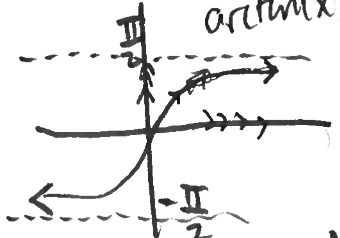
⇓

order-preserving

⇓

if $b>a$, then $\tan(b)>\tan(a)$

ancient memory



Need to enforce

$$R_N < 0.0001$$

To do it,

$$\boxed{R_N < \frac{\pi}{2} - \arctan(N) < 0.0001}$$

enforce

$$\arctan(N) > \frac{\pi}{2} - 0.0001$$

⇓ plug into tangent

$$N > \tan\left(\frac{\pi}{2} - 0.0001\right) \approx 9999.99997$$

calculator

So, $N = 10000$ will work!

Using $N = 10000$, estimate

$$S_N = S_{10000} \overset{\text{def}}{=} \sum_{k=1}^{10000} \frac{1}{1+k^2}$$

$$\overset{\text{computer}}{\approx} \boxed{1.0765}74$$

SAME (BOX)

different

Compare to "actual" ∞ sum:

$$\text{Computer: } \sum_{k=1}^{\infty} \frac{1}{k^2+1} \approx \boxed{1.0766}74$$

Comparison tests

We know two easy classes of series:

geometric: $\sum^{\infty} r^k$ ← variable power, ← constant r

$|r|<1 \Rightarrow$ conv; $|r| \geq 1$ diverges

p-series: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ ~ converge when $p>1$

diverge $p \leq 1$

Ex: Does $\sum_{k=0}^{\infty} \frac{1}{1+k^2}$ converge?

Answer: "Clearly"

$$k^2+1 > k^2$$

$$\big\downarrow \text{ reciprocal}$$

$$1 > \frac{k^2}{k^2+1} \qquad \frac{1}{k^2+1} < \frac{1}{k^2}$$

$$\frac{1}{k^2} > \frac{1}{k^2+1} \qquad \sum_{k=0}^{\infty} \frac{1}{k^2+1} < \underbrace{\sum_{k=0}^{\infty} \frac{1}{k^2}}_{\text{this converges by p-series}}$$

$\frac{1}{4} < \frac{1}{2}$

$\frac{2}{4} < 1$

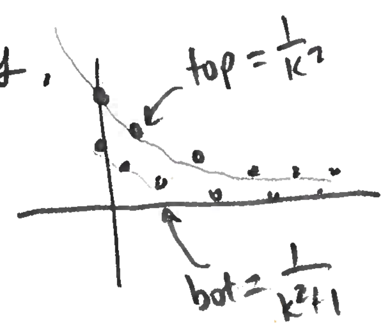
$2 < 4$

$\frac{1}{2} > \frac{1}{4}$

So we have that

$0 < \sum_{k=0}^{\infty} \frac{1}{k^2+1}$ is less than a converging series

So by comparison test, it converges.



Ex: Does $\sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}}$ conv or div?

$$k - \frac{1}{2} < k$$

$$\downarrow$$

$$\frac{1}{k-\frac{1}{2}} > \frac{1}{k}$$

$$\downarrow$$

$$\sum \frac{1}{k-\frac{1}{2}} > \underbrace{\sum \frac{1}{k}}_{\text{series diverges! it is harmonic series}}$$

to ∞

So we have

$\sum \frac{1}{k-\frac{1}{2}}$ is bigger than a series that diverges to ∞

Therefore $\sum^{\infty} \frac{1}{k-\frac{1}{2}}$ diverges.

Theorem :

(i) Suppose $0 \le a_k \le b_k$. If $\sum^{\infty} b_k$ converges, then so does $\sum^{\infty} a_k$.

(ii) Suppose $a_k \ge b_k \ge 0$. If $\sum^{\infty} b_k$ diverges, then so does $\sum^{\infty} a_k$.

Ex : (won't help)

Conv or div? $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k+1/2}$

$a_k \le b_k$

Try: $k + \frac{1}{2} > k$

$$\underbrace{\frac{1}{k+\frac{1}{2}}}_{a_k} < \underbrace{\frac{1}{k}}_{b_k}$$

(in (i) of thm)

$$\sum \frac{1}{k+1/2} < \underbrace{\sum \frac{1}{k}}_{\text{diverges}}$$

BUT

$\sum b_k$ diverges (condition from (ii))

$\Downarrow$

mismatch & can't proceed!