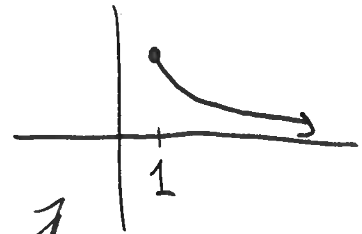
Ex: Use integral test to determine if

$$\sum_{k=1}^{\infty} \frac{3}{7^{\ln(k)}}$$



Consider $f(x) = \dfrac{3}{7^{\ln(x)}}$

✓ i) continuous

✓ ii) decreasing

So compute

✓ $\ln(a^{b}) = b\ln(a)$

$$\int_1^{\infty} \frac{3}{7^{\ln(x)}}\,dx = 3 \lim_{b\to\infty} \int_1^b x^{-\ln(7)}\,dx$$

$$= 3 \lim_{b\to\infty} \left. \frac{x^{1-\ln(7)}}{1-\ln(7)} \right|_1^b$$

$$= \frac{3}{1-\ln(7)} \lim_{b\to\infty} \left[b^{1-\ln(7)} - \cancel{1^{1-\ln(7)}}^{=1} \right]$$

∞? 0?

$$= \frac{3}{1-\ln(7)} \lim_{b\to\infty} \left[\cancel{b^{-0.946}} - 1 \right]$$

$\to 0$ as $b \to \infty$

$\frac{1}{b^{0.946}}$

$$= \frac{3}{\ln(7)-1}$$

$$7^{\ln(x)} = e^{\ln(7^{\ln(x)})} = e^{\ln(x)\ln(7)} = \cancel{e}^{\cancel{\ln}(x^{\ln(7)})} = x^{\ln(7)}$$

plain old #

useful trick

5^x

realization allowed us to $\int$

$\Rightarrow$ integral converges

$\Rightarrow$ series $\displaystyle\sum_{k=1}^{\infty} \frac{3}{7^{\ln(k)}}$ converges

Fact is, most series you can write do **NOT** have a "nice" sum that can be perfectly expressed in terms of "nice" things.

Theorem: Suppose $\sum^{\infty} a_k$ converges as has positive terms. Suppose also there is a function f that is continuous, decreasing, and $a_k = f(k)$. Let $S_N = \sum^{N} a_k$

Then,

$$\underbrace{S_N + \int_{N+1}^{\infty} f(x)\,dx}_{\text{lower bound}} < \sum^{\infty} a_k < \underbrace{S_N + \int_{N}^{\infty} f(x)\,dx}_{\text{upper bound}}$$

In other words, $R_N = \left(\sum^{\infty} a_k\right) - S_N = \underbrace{\sum_{k=N+1}^{\infty} a_k}_{\text{tail of series}}$,

then

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$$

R_N called residual error

FACTS

Recall $S_N \to \sum^{\infty} a_k$ as $N \to \infty$

So, $R_N \to 0$ as $N \to \infty$

Ex: (Apéry's constant $\sum_{k=1}^{\infty} \frac{1}{k^3}$)

Find a value of N so that $R_N < 0.001$

guarantees S_N is correct value for a few decimal points

Once N found, compute S_N for that value of N.

Soln: By series estimation thm,

$$R_N < \int_N^{\infty} \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty} \int_N^b x^{-3}\,dx$$

$$= \lim_{b\to\infty} \frac{x^{-2}}{-2}\Big|_N^b = -\frac{1}{2} \lim_{b\to\infty}\left(\cancelto{0}{\frac{1}{b^2}} - \frac{1}{N^2}\right)$$

$$= \frac{1}{2N^2}$$

So we have

$$\boxed{R_N < \frac{1}{2N^2} \overset{\text{enforce}}{<} 0.001}$$

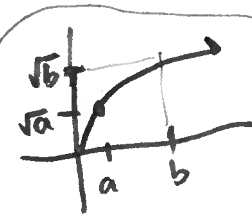
I control N

$$\frac{1}{0.001} < 2N^2 \longrightarrow 500 < N^2$$

$\downarrow$ $\sqrt{\ }$ preserves order

$$22.36 = \sqrt{500} < N$$

So take $N = 23$. Now compute $S_{23} = \sum_{k=1}^{23} \frac{1}{k^3} = 1.2012$



FACT $\sqrt{\ }$ function is "order-preserving"

meaning

$a < b \longrightarrow \sqrt{a} < \sqrt{b}$

$R_N < 0.001$

<u>Ex</u>: Same but $\displaystyle\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$, $\epsilon = 0.001$

$$f(x) = \frac{\ln(x)}{x^2} \rightsquigarrow \begin{matrix} \text{i) ctn} \\ \text{ii) dec} \end{matrix} \checkmark$$

So compute

$$R_N < \int_N^{\infty} \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty} \int_N^b \ln(x)\,x^{-2}\,dx$$

$-x^{-1}$

$u = \ln(x)$ $dv = x^{-2}dx$

$du = \frac{1}{x}dx$ $v = -x^{-1}$

parts

$$= \lim_{b\to\infty} -\frac{\ln(x)}{x}\Big|_N^b + \int_N^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} \Big(\underbrace{-\frac{\ln(b)}{b}}_{\to\, 0} + \frac{\ln(N)}{N}\Big) + \Big(\underbrace{-\frac{1}{b}}_{\to\, 0} + \frac{1}{N}\Big)$$

(L'Hôpital)

$$= \frac{\ln(N)}{N} + \frac{1}{N}$$

So we have

$$\boxed{R_N < \underbrace{\frac{\ln(N)}{N} + \frac{1}{N}}\ \overset{\text{enforce}}{<}\ \underbrace{0.001}}$$

can't solve for N

can use $N = 22595$

$\Downarrow$

$S_N = \_\_\_$