Turns out

$$\zeta(z) \overset{def}{=} \sum_{k=1}^{\infty} \frac{1}{k^z}$$

converges by p-series test for any $z>1$.

Solved Basel problem from 1650

$\zeta(2)$ ~ was shown by Euler to equal $\pi^2/6$ in 1734

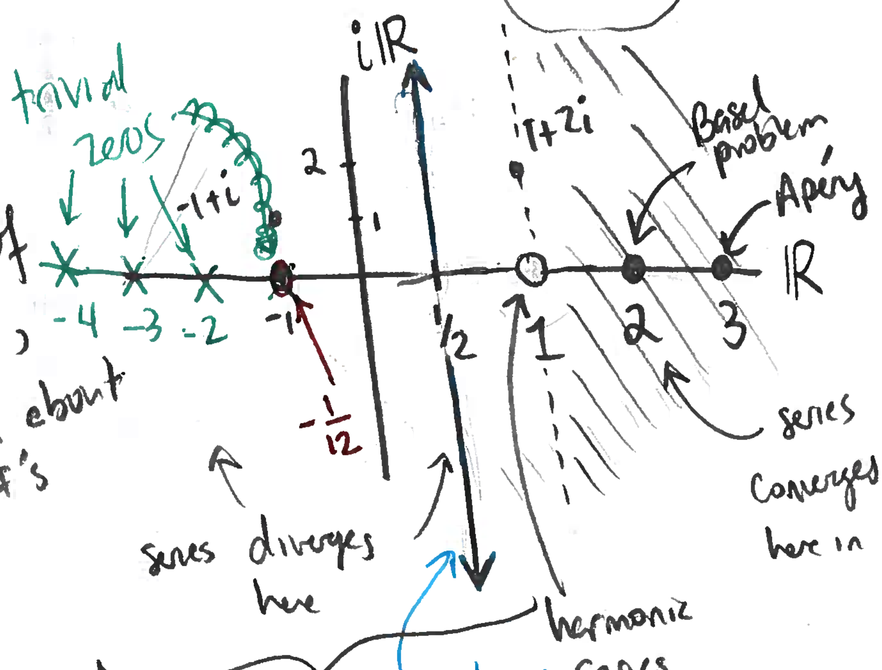
In the famous paper "On the number of primes less than a given magnitude" from B. Riemann in 1859 ~ the ζ function was allowed to take $z=a+bi$ ~ complex argument.

$i=\sqrt{-1}$

↓

$i^2=-1$

if you look at solutions of $\zeta(z)=0$, you learn about prime #'s



complex plane ℂ

$1+2i$

$-1+i$

requires "analytic continuation"

answer to this is worth $1,000,000

→ Riemann hypothesis: All nontrivial solutions to $\zeta(z)=0$ on critical line?
(open problem) Are

Recall

$$\text{"}\underbrace{1+2+3+\dots}_{\zeta(-1)} = -\frac{1}{12}\text{"}$$

(via analytic continuation)

Naively:

$$\zeta(-1) = \sum_{k=1}^{\infty} \frac{1}{k^{-1}} = \sum_{k=1}^{\infty} k$$

diverges

Few more results:

(1921) Hardy + Littlewood: ∞-many of zeros lie on critical line

(1942) Selberg: a positive proportion of zeros lie on crit line

(1974) Levinson: at least $1/3$ of zeros lie on line

(1989) Conrey: at least $2/5$ lie on line

Ex: Conv or diverge?

$$\sum_{k=1}^{\infty} \frac{8\ln(k)}{k^6}$$

Can we use $\int$ test? $f(x) = \frac{\ln(x)}{x^6}$

✓ i) continuous on $[1,\infty)$?

✓ ii) decreasing on $[1,\infty)$?

~~no~~ but it is on $[2,\infty)$ which is good enough

So compute

$$\int_1^{\infty} \frac{\ln(x)}{x^6}\,dx \overset{\text{(parts)}}{=} \lim_{b\to\infty} \int_1^b \ln(x)\, x^{-6}\,dx$$

$\cancel{u = \ln x}$

$\cancel{du = \frac{1}{x}dx}$

$u = \ln x \quad dv = x^{-6}$

$du = \frac{1}{x}dx \quad v = \frac{x^{-5}}{-5}$

$$= \lim_{b\to\infty} \left[-\frac{1}{5}\frac{\ln(x)}{x^5}\Big|_1^b + \frac{1}{5}\int_1^b x^{-6}\,dx \right]$$

$$= \lim_{b\to\infty} \left\{ \left[-\frac{1}{5}\frac{\ln(b)}{b^5} + 0 \right] + \frac{x^{-5}}{-25}\Big|_1^b \right\}$$

$$\lim_{b\to\infty} \frac{\ln(b)}{b^5} = \frac{\infty}{\infty}$$

$$\overset{LH}{=} \lim_{b\to\infty} \frac{1}{5b^5} = 0$$

$$= \lim_{b\to\infty} \left(\underbrace{-\frac{1}{5}\frac{\ln(b)}{b^5}}_{0} - \underbrace{\frac{1}{25b^5}}_{0} + \frac{1}{25} \right)$$

$\ln(1) = 0$

$$= \frac{1}{25}$$

$\Rightarrow$ integral converges

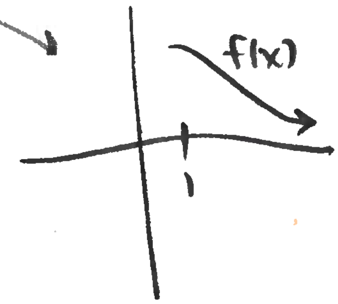
$\Rightarrow$ by $\int$ test, series $\sum_{k=1}^{\infty} \frac{8\ln(k)}{k^6}$ converges

Ex: Does $\sum_{k=1}^{\infty} k e^{-k}$ conv or diverge?

$f(x) = xe^{-x}$ on $[1,\infty)$:

✓ i) continuous?

✓ ii) decreasing?



⇒ So compute

$$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx$$

anti-deriv is $-e^{-x}$

$u = x \quad dv = e^{-x}$

$du = dx \quad v = -e^{-x}$

$$= \lim_{b\to\infty}\left[-xe^{-x}\Big|_1^b + \int_1^b e^{-x}\,dx\right]$$

$$= \lim_{b\to\infty}\left[\left(\underbrace{\cancel{-be^{-b}}}_{0} + e^{-1}\right) + \left(\cancel{-e^{-b}} + e^{-1}\right)\right]$$

$\frac{1}{e^b} \to 0$ as $b\to\infty$

"indet"

$= \infty e^{-\infty}$

$\overset{L'H}{=} \lim_{b\to\infty} \dfrac{\frac{d}{db}b}{\frac{d}{db}e^b}$

$= -\lim_{b\to\infty} \frac{1}{e^b} = 0$

$$= \frac{2}{e}$$

⇒ integral converges

⇒ by integral test, series $\sum_{k=1}^{\infty} ke^{-k}$ conv.