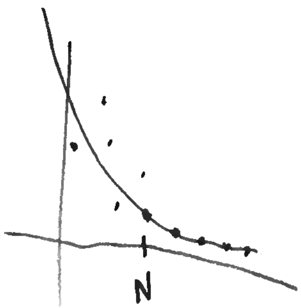
<u>Theorem</u> (integral test for convergence + divergence of series) Suppose $\sum_{k=1}^{\infty} a_k$ has <u>positive</u> terms a_k.

Suppose there is a function f such that



✓ i) f is continuous

✓ ii) f is decreasing, and

✓ iii) $f(k) = a_k$ for all $k \geq N$ ← some large value

THEN,

$\sum^{\infty} a_k$ and $\int^{\infty} f(x)\,dx$ either both converge or both diverge.

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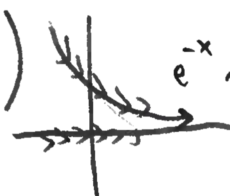
conv or div?

<u>Ex</u>: $\sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^{k}$ ⤳ Can we use $\int$ test?

this is a geometric series ($\Rightarrow$ converge b/c $0 < \frac{1}{e} < 1$)

✓ i) $f(x) = \left(\frac{1}{e}\right)^x = \frac{1}{e^x} = e^{-x}$ is continuous!

✓ ii)  e^{-x} ~ decreasing!

So compute

$$\int_1^{\infty} \left(\frac{1}{e}\right)^x dx = \int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= -\lim_{b\to\infty} e^{-x}\Big|_1^b = -\lim_{b\to\infty}\left(\cancel{e^{-b}}^{\,0} - e^{-1}\right)$$

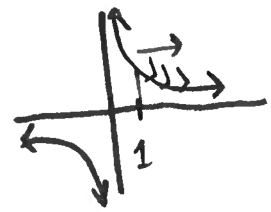
$$= e^{-1} = \frac{1}{e}$$

$\Rightarrow$ the integral converges

$\Rightarrow$ by integral test, can conclude sum also converges!

Ex: Converge or diverge?

$\sum_{k=1}^{\infty} \frac{1}{k^3}$ ~ here we let $f(x) = \frac{1}{x^3}$



and check

✓ i) f is continuous ✓ (except at $x=0$)

✓ ii) f is decreasing ✓

ok since we will only $\int$ on $[1,\infty)$

So, compute

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_1^b x^{-3}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^b$$

$$= -\frac{1}{2} \lim_{b\to\infty} \left[\cancel{\frac{1}{b^2}}^{\,0} - 1\right]$$

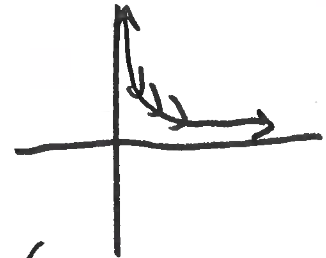
$$= 1/2$$

$\Rightarrow$ integral converges!

$\Rightarrow$ by $\int$-test, the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ also converges

Ex: Conv or div?

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k-1}} \sim f(x) = \frac{1}{\sqrt{2x-1}}$$



✓ i) f is continuous ✓

✓ ii) f is decreasing

So compute

$$\int_1^{\infty} \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty} \int_1^b (2x-1)^{-1/2}\,dx$$

$u = 2x-1$

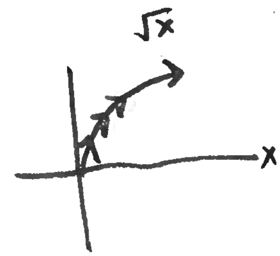
$\frac{1}{2}du = dx$

$x=1 \to u = 2-1 = 1$

$x=b \to u = 2b-1$

$$= \lim_{b\to\infty} \int_1^{2b-1} \frac{1}{2} u^{-1/2}\,du$$

$$= \frac{1}{2}\lim_{b\to\infty} \left.\frac{u^{1/2}}{1/2}\right|_{u=1}^{u=2b-1}$$



$$= \frac{2}{2}\lim_{b\to\infty} \left((2b-1)^{1/2} - 1 \right)$$

∞

"=" ∞

$\Rightarrow$ integral diverges

$\Rightarrow$ series diverges

p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$ where p is a positive real # $\sim$ $f(x) = \frac{1}{x^p}$

✓ i) f continuous ✓

✓ ii) f decreasing

Consider

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$\boxed{p \neq 1}$

$$= \lim_{b\to\infty} \left[x^{-p+1} \right]_1^b$$

$$= \lim_{b\to\infty} \left(b^{-p+1} - 1 \right)$$

$\underline{p=2}$

$$\lim_{b\to\infty} b^{-2+1} = \lim_{b\to\infty} \frac{1}{b} = 0 \checkmark$$

$\underline{p=1}$

nonsense $\sim$ here antideriv is $\ln(x)$ and so will diverge

$\underline{p=0.5}$

$$\lim_{b\to\infty} b^{-0.5+1} = \lim_{b\to\infty} \sqrt{b} = \infty$$

FACT: * if "$-p+1$" is negative, then the limit will ~~exist~~

$-p+1 < 0$

$1 < p$

* if "$-p+1$" is positive, then limit DNE

$-p+1 > 0$

$1 > p$

* if "$-p+1$" is zero, then a log occurs + limit DNE

So we obtain

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} 0, & p > 1 \Rightarrow \int \text{conv} \Rightarrow \sum \text{conv} \\ \text{DNE}, & p \leq 1 \Rightarrow \int \text{div} \Rightarrow \sum \text{div} \end{cases}$$

FACT: p-series is just the famous

Riemann zeta function

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$$

Can be shown that...

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90} \qquad \zeta(6) = \pi/945$$

BUT

$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$ has "unknown" value in closed form

shown irrat'l by Apery in 1978

YR 2000: Rivoal showed ∞-many $\zeta(\text{odd})$ are irrat'l

YR 2001: Zudilin showed that one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ is irrat'l