Ex: Find partial sum S_N and take its limit to compute the value of telescoping series

$$\sum_{k=0}^{\infty} \frac{1}{(k+4)(k+5)}$$

Partial fractions:

$$\frac{1}{(k+4)(k+5)} = \frac{A}{k+4} + \frac{B}{k+5}$$

$$1 = (A)(k+5) + (B)(k+4)$$

$$0k + 1 = (A+B)k + (5A+4B)$$

$$\Rightarrow \begin{cases} A+B=0 \\ 5A+4B=1 \end{cases} \rightarrow \boxed{A=-B} \rightarrow \boxed{A=1}$$

$$-5B+4B=1 \rightarrow \boxed{B=-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{(k+4)(k+5)} \overset{\text{p.f.}}{=} \sum_{k=0}^{\infty} \left(\frac{1}{k+4} - \frac{1}{k+5} \right)$$

So, $$S_N \overset{\text{def}}{=} \sum_{k=0}^{N} \left(\frac{1}{k+4} - \frac{1}{k+5} \right) = \left(\frac{1}{4} - \cancel{\frac{1}{5}} \right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} \right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{7}} \right)$$

$$+ \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{8}} \right) + \dots + \left(\cancel{\frac{1}{N+4}} - \frac{1}{N+5} \right)$$

No "N" — unaffected

not cancel!

$$= \frac{1}{4} - \frac{1}{N+5}$$

$$\sum_{k=0}^{\infty} \frac{1}{(k+4)(k+5)} \overset{\text{def}}{=} \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{4} - \cancel{\frac{1}{N+5}}^{\to 0} \right) = \frac{1}{4}$$

Test for divergence

FACT: if $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

Why? If $\lim_{k\to\infty} a_k = 2$ (for ex)

we sum these values!

Series $\sum_{k=0}^{\infty} a_k = \ldots\ldots + \underbrace{2+2+2}_{\text{"ish"}} + \ldots \to \infty$

CAREFUL

This fact does not "go other way" ~ it is

the case that **NOT**

"if $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=0}^{\infty} a_k$ converges"

Why? harmonic series: $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges — counterexample

but here $a_k = \frac{1}{k} \xrightarrow{k\to\infty} 0$

Theorem: if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges"

"test for divergence"

(could mean finite value, ∞, $-\infty$, oscillate)

Examples: Does test for div. tell you anything about...

a) $\sum_{k=1}^{\infty} \frac{k}{3k-1}$ ~ $\lim_{k\to\infty} \underbrace{\frac{k}{3k-1}}_{a_k} = \frac{1}{3} \neq 0 \overset{\text{TFD}}{\Rightarrow}$ series diverges

b) $\sum_{k=1}^{\infty} \frac{1}{k^3}$ ~ $\lim_{k\to\infty} \frac{1}{k^3} = 0$ No conclusion (might converge, might diverge)

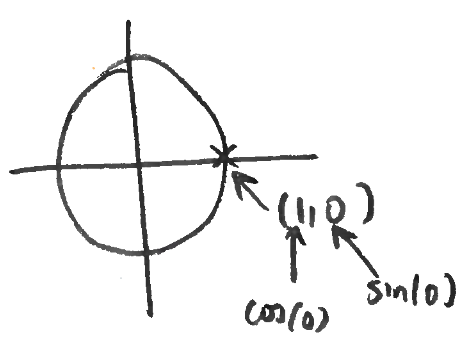
c) $\sum_{k=1}^{\infty} e^{\frac{1}{k^2}}$ ~ $\lim_{k\to\infty} e^{\frac{1}{k^2}} = e^{\lim_{k\to\infty}\frac{1}{k^2}} = e^0 = 1 \neq 0$

$\overset{\text{TFD}}{\Rightarrow}$ series diverges

d) $\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^2}\right)$ ~ $\lim_{k\to\infty} \cos\left(\frac{1}{k^2}\right) = \cos\left(\lim_{k\to\infty}\frac{1}{k^2}\right)$

$= \cos(0) = 1 \neq 0$

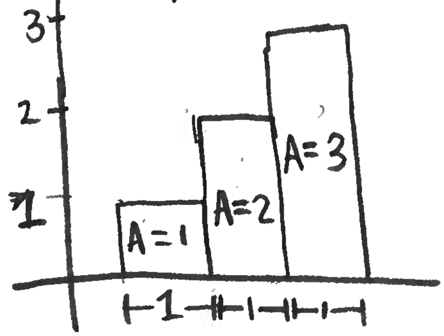
$\overset{\text{TFD}}{\Rightarrow}$ series diverges



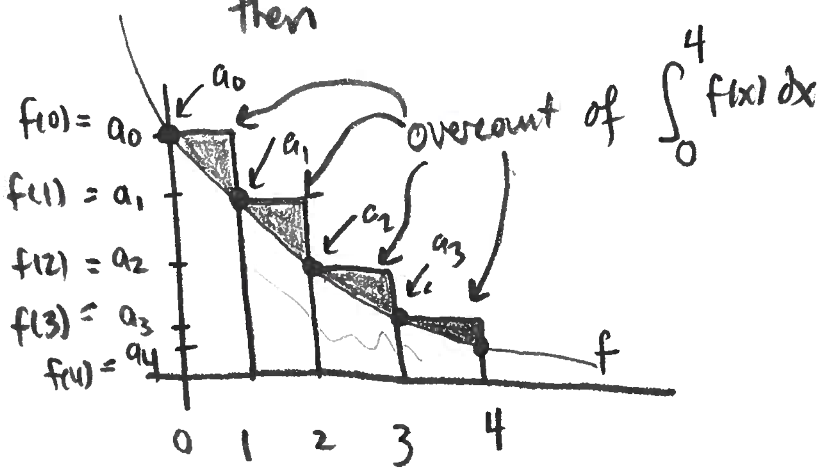
Integral test

Notice: $1+2+3=6$

can be interpreted in terms an area:



Useful to see: if $a_k = f(k)$ for some function f, then



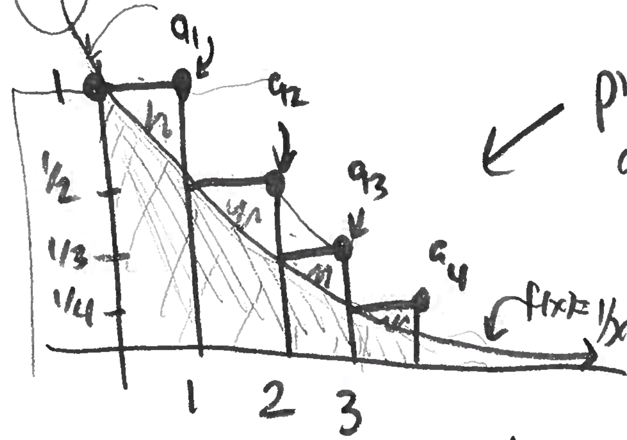
Conclusion: in this picture,

$$\sum_{k=0}^{\cancel{3}4} a_k > \int_0^4 f(x)\,dx$$

So: $$\sum_{k=0}^{N} a_k > \int_0^N f(x)\,dx \quad \xRightarrow{\text{take } \lim_{N\to\infty}} \quad \sum_{k=0}^{\infty} a_k > \underbrace{\int_0^{\infty} f(x)\,dx}_{\text{compute this}}$$

Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$



picture is a little off

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \underbrace{\int^{\infty} \frac{1}{x}\,dx}_{\text{compute!}} = \lim_{N\to\infty} \int_1^N \frac{1}{x}\,dx$$

$$= \lim_{N\to\infty} \ln(N) - \cancelto{0}{\ln(1)}$$

$$= \infty$$

Conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \infty \sim \text{i.e. } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$