Geometric series

$(\star)\quad \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ ONLY if $|r|<1$

$(\star\star)\quad \sum_{k=a}^{\infty} r^k = \frac{r^a}{1-r}$ ONLY if $|r|<1$

[Note: A series is geometric if it CAN BE written in form $(\star)$]

Ex: $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{k+2}$

Way 1

Realize

$\left(\frac{2}{3}\right)^{k+2} = \left(\frac{2}{3}\right)^2 \left(\frac{2}{3}\right)^k$ $\quad |r| = \left|\frac{2}{3}\right| < 1$

So,

$$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{k+2} = \left(\frac{2}{3}\right)^2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

$$\overset{(\star)}{=} \left(\frac{2}{3}\right)^2 \frac{1}{1-2/3}$$

$$= \frac{4}{9} \cdot \frac{1}{1/3}$$

$$= \frac{4}{9} \cdot \frac{3}{1} = \frac{4}{3}$$

Way 2

Reindex:

$$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{k+2} = \sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k$$

$$\overset{(\star\star)}{=} \frac{(2/3)^2}{1-2/3}$$

$$= \frac{(4/9)}{(1/3)} = \frac{4}{3}$$

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^{k+1}}{4^{k-1}} = \sum_{k=0}^{\infty} \frac{(-3)^{k+2}}{4^k} = 9\sum_{k=0}^{\infty}\left(\frac{-3}{4}\right)^k$

$(-3)^{k+2} = (-3)^2(-3)^k$

$\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$

(2)

$r = \frac{-3}{4}$ ✓ (★)

$|r| = \frac{3}{4} < 1$

$= 9\dfrac{1}{1-(-3/4)}$

$= \dfrac{9}{7/4} = \dfrac{36}{7}$

different route

$\dfrac{1}{4^{k-1}} = \left(\dfrac{4}{4}\right)\cdot\dfrac{1}{4^{k-1}}$

$= \dfrac{4}{4^k}$

use (★★)

Ex: $\displaystyle\sum_{k=1}^{\infty} e^{2k} = \sum_{k=1}^{\infty} (e^2)^k \overset{(★★)}{=} \cancel{\frac{e^2}{1-e^2}}$

e is a number

$e^{2k} = (e^2)^k$

can't use (★★) because

NOT CORRECT ~ this series does not converge, because

$r = e^2 \approx (2.71)^2 > 1$

Ex: $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{2}{5}\right)^{n-1} = \sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^{n} \overset{(★)}{=} \frac{1}{1-(-2/5)} = \frac{5}{7}$

raise by 1

lower by 1

$r = \frac{-2}{5}$

$\to |r| < 1$ ✓

EX: FACT: $0.\underbrace{9999\ldots}_{\text{infinitely many 9's}} = 1$

Ways to see:

① (Arithmetic)

$\frac{1}{3} = 0.333\ldots$

$\frac{2}{3} = 0.6666\ldots$

$1 = \frac{3}{3} = 0.9999\ldots$

② (Algebraic)

$x = 0.999\ldots$

$10x = 9.999\ldots$

$10x - x = 9.\cancel{999}\ldots - 0.\cancel{999}\ldots$

$9x = 9$

$x = 1$

③ Geometric series

$$0.999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$$

$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \ldots$$

$$= 9\cdot 10^{-1} + 9\cdot 10^{-2} + 9\cdot 10^{-3} + 9\cdot 10^{-4} + \ldots$$

$$= \sum_{k=1}^{\infty} 9\cdot 10^{-k} = 9\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k \overset{(**)}{=} 9\,\frac{\left(\frac{1}{10}\right)^1}{\left(1-\frac{1}{10}\right)}$$

$r = \frac{1}{10}$

$|r| < 1$

$$= \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

$\boxed{2 = \frac{4}{2} = \frac{8}{4}}$

Telescoping series

Yesterday $\rightsquigarrow$ we saw $\displaystyle\sum_{K=1}^{\infty} \frac{1}{K(K+1)}$

had a partial sum $S_N = \dfrac{N}{N+1}$.

Turns out this series is a "telescoping" series.

Partial fractions:

$$\frac{1}{K(n+1)} = \frac{A}{K} + \frac{B}{K+1}$$

$$1 = (A)(K+1) + (B)(K)$$

$$0K + 1 = (A+B)K + (A) \Rightarrow \begin{cases} \boxed{A=1} \to \boxed{B=-1} \\ A+B=0 \to B=-A \end{cases}$$

So,

$$\sum_{K=1}^{\infty} \frac{1}{K(K+1)} = \sum_{K=1}^{\infty} \left(\frac{1}{K} - \frac{1}{K+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$$

Telescoping: partial sum

cancels cancels cancels cancels

$$S_N = \sum_{K=1}^{N} \left(\frac{1}{K} - \frac{1}{K+1}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \dots + \left(\cancel{\tfrac{1}{N}} - \frac{1}{N+1}\right)$$

not be canceled

$$= 1 - \frac{1}{N+1}$$

So from here, easy to see

$$\sum_{K=1}^{\infty} \frac{1}{K(K+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \underbrace{\frac{1}{N+1}}_{\to 0} = 1$$

def of series