$$\sum_{k=1}^{\infty} a_k = \underbrace{a_1 + a_2 + a_3 + a_4 + \dots}_{\text{What does this mean?}}$$

"capital" "sigma"

(σ)
lowercase

Define it as

$$\sum_{k=1}^{\infty} a_k = \lim_{N\to\infty} \underbrace{\sum_{k=1}^{N} a_k}_{\text{"partial sum" } S_N} = \lim_{N\to\infty} S_N$$

If $\lim_{N\to\infty} S_N$ exists, (as a real number) then we say that the series converges to a number $S = \lim_{N\to\infty} S_N$. Otherwise we say it diverges.

Examples: a) $\sum_{k=1}^{\infty} \frac{k}{k+1}$ ~ converge or diverge?

$$S_1 = \sum_{k=1}^{1} \frac{k}{k+1} = \frac{1}{2} \qquad = 1/2$$

$$S_2 = \sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} \qquad > 2/2$$

$$S_3 = \sum_{k=1}^{3} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \qquad > 3/2$$

$$S_4 = \sum_{k=1}^{4} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \qquad > 4/2$$

$>1/2$ $>1/2$ $>1/2$

$$\frac{\infty}{\infty}$$

$$\lim_{k\to\infty} \frac{k}{k+1} \overset{L'H}{=} \lim_{k\to\infty} \frac{1}{1} = 1$$

Generally, $S_N > N/2$, but $\lim_{N\to\infty} N/2 = \infty$ ($\therefore$ series diverges)

b) (Grandi's series)

$$\sum_{k=0}^{\infty} (-1)^k$$

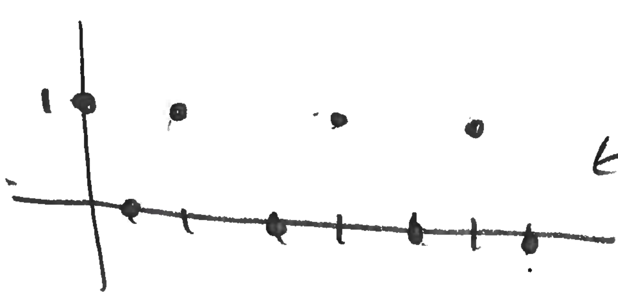
$$S_0 = \sum_{k=0}^{0} (-1)^k = (-1)^0 = 1$$

$$S_1 = \sum_{k=0}^{1} (-1)^k = (-1)^0 + (-1)^1 = 1 - 1 = 0$$

$$S_2 = \sum_{k=0}^{2} (-1)^k = (-1)^0 + (-1)^1 + (-1)^2 = \cancel{1-1} + 1 = 1$$

$$S_3 = \sum_{k=0}^{3} (-1)^k = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 = 1 - 1 + 1 - 1 = 0$$

$$S_N = \begin{cases} 1, & N = 0, 2, 4, 6, 8, \ldots \\ 0, & N = 1, 3, 5, 7, 9, \ldots \end{cases}$$



does not converge! heights do not get close to a single value!

In sense of our text, Grandi's series does not converge to a value!

c) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

$\sum^{\infty} \frac{1}{☺(☺+1)}$ 3

☺ $=1$

$S_1 = \frac{1}{1\cdot 2} = 1/2$

$S_2 = \sum_{k=1}^{2} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = 2/3$

$S_3 = \sum_{k=1}^{3} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = 3/4$

$S_4 = \sum_{k=1}^{4} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = 4/5$

$\vdots$

$\frac{1}{2} + \frac{1}{6}$

$\frac{3+1}{6} = \frac{4}{6}$

$\frac{2}{3} + \frac{1}{12}$

$\frac{8+1}{12} = \frac{9}{12} = \frac{3}{4}$

$\frac{3}{4} + \frac{1}{20} = \frac{15+1}{20}$

$= \frac{16}{20} = \frac{8}{10} = \frac{4}{5}$

The pattern seems to be

$$S_N = \frac{N}{N+1}$$

So,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \overset{\text{def}}{=} \lim_{N\to\infty} S_N$$

$$= \lim_{N\to\infty} \frac{N}{N+1}$$

$$= 1$$

"telescoping"

Algebraic rules for infinite series

If $\sum a_n$ and $\sum b_n$ converge, then

i) $\sum (a_n \pm b_n) = \left(\sum a_n\right) \pm \left(\sum b_n\right)$

ii) if α is a number, then

$$\sum \alpha a_n = \alpha \sum a_n$$

in linear algebra, you would call these properties a "linear transform"

Reindexing a sum

The idea of expressing any sum w/ different start value

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8

Ex: $\displaystyle\sum_{k=3}^{8} \frac{k}{e^k} = \frac{3}{e^3} + \frac{4}{e^4} + \frac{5}{e^5} + \frac{6}{e^6} + \frac{7}{e^7} + \frac{8}{e^8}$

Reindex so it starts at $k=0$:

$$\sum_{\tilde{k}=0} \frac{\tilde{k}+3}{e^{\tilde{k}+3}} = \sum_{\tilde{k}=0}^{5} \frac{\tilde{k}+3}{e^{\tilde{k}+3}} = \frac{3}{e^3} + \frac{4}{e^4} + \frac{5}{e^5} + \frac{6}{e^6} + \frac{7}{e^7} + \frac{8}{e^8}$$

Value at $\tilde{k}=0$ should match value at $k=3$

(we had to add 3 to index var everywhere)

Reindex so it starts w/ $k=1$

$$\sum_{k=1}^{6} \frac{k+2}{e^{k+2}} = \frac{3}{e^3} + \dots + \frac{8}{e^8}$$