Ex: $\displaystyle\int \frac{x^3+1}{x^2+2x+1}\,dx$

Here, $P(x) = x^3+1$, $Q(x) = x^2+2x+1$

$\deg(P(x)) = 3 \geq 2 = \deg(Q(x))$

Step 1

$$\begin{array}{r} x-2 \qquad R(3x+3) \\ x^2+2x+1 \overline{\big)\, x^3+1 \qquad\qquad} \\ -(x^3+2x^2+x) \\ \hline -\cancel{2x^2}-x+1 \\ -(-\cancel{2x^2}-4x-2) \\ \hline 3x+3 \end{array}$$

This means

$$\frac{x^3+1}{x^2+2x+1} = \underbrace{x-2}_{\text{easy to } \int} + \underbrace{\frac{3x+3}{x^2+2x+1}}_{\text{require p.f.}}$$

Step 2 $\left(\text{for } \frac{3x+3}{x^2+2x+1}\right)$

$x^2+2x+1 = (x+1)^2$ so $\displaystyle\frac{3x+3}{x^2+2x+1} = \frac{3x+3}{(x+1)^2} = \frac{3\cancel{(x+1)}}{(x+1)^{\cancel{2}}}$

BIG COINCIDENCE → we won't need pf → $= \dfrac{3}{x+1}$

So, $\quad a\ln(x) = \ln(x^a)$

$$\int \frac{x^3+1}{x^2+2x+1}\,dx = \int x-2\,dx + \int \frac{3x+3}{x^2+2x+1}\,dx$$

$u = x+1$

$3\int \frac{1}{u}\,du$

$$= \frac{x^2}{2} - 2x + \int \frac{3}{x+1}\,dx$$

$$= \frac{x^2}{2} - 2x + 3\ln(x+1) + C$$

Ex: $\displaystyle\int \frac{x^3+2}{x^2+4x+3}\,dx$

Step 1) $\deg(x^3+2) = 3 \geq 2 = \deg(x^2+4x+3)$

$$\begin{array}{r l}
 & x-4 \quad R\ 13x+14 \\
x^2+4x+3 \,\big) & \cancel{x^3} + 2 \\
 & -(\cancel{x^3}+4x^2+3x) \\
\hline
 & -\cancel{4x^2}-3x+2 \\
 & -(-\cancel{4x^2}-16x-12) \\
\hline
 & 13x+14
\end{array}$$

Therefore,

$$\frac{x^3+2}{x^2+4x+3} = \underbrace{x-4}_{\text{easy to } \int} + \underbrace{\frac{13x+14}{x^2+4x+3}}_{\text{P.F.}}$$

Step 2 (for $\frac{13x+14}{x^2+4x+3}$)

Factor denom:

$$\frac{13x+14}{x^2+4x+3} = \frac{13x+14}{(x+3)(x+1)}$$

Step 3 :

$$\frac{13x+14}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

$\downarrow$ mult by denom

$$13x+14 = (A)(x+1) + (B)(x+3)$$

$$= (A+B)x + (A+3B)$$

$$\Rightarrow \begin{cases} A+B = 13 & \text{(i)} \\ A+3B = 14 & \text{(ii)} \end{cases}$$

(i) $\rightarrow$ $\boxed{A = 13 - B}$

(ii) $\rightarrow$ $(13-B)+3B = 14$

$2B = 1$

$\boxed{B = 1/2}$

Step 4

$$\boxed{\begin{aligned} A &= 13 - B \\ &= 13 - 1/2 \\ &= \frac{25}{2} \end{aligned}}$$

Step 5

$$\int \frac{x^3+2}{x^2+4x+3}\,dx = \int x-4\,dx + \int \frac{13x+44}{x^2+4x+3}\,dx$$

$$= \int x-4\,dx + \frac{25}{2}\int \frac{1}{x+3}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2} - 4x + \frac{25}{2}\ln(x+3) + \frac{1}{2}\ln(x+1) + C$$

Ex: $\displaystyle\int \frac{x^2+1}{x^3+6x^2+9x}\,dx$

$x^2+3 \leftarrow$ irreduc quad

$x^2-3 \leftarrow$ two linears

Step 1 no long div

Step 2: $\displaystyle\frac{x^2+1}{x(x^2+6x+9)} = \frac{x^2+1}{x(x+3)^2}$ ← power of linear

Step 3: $\displaystyle\frac{x^2+1}{x(x+3)^2} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

↓ mult. by common denom

$$1x^2+0x+1 = A\underbrace{(x+3)^2}_{x^2+6x+9} + B(x^2+3x) + Cx$$

$$= (A+B)x^2 + (6A+3B+C)x + (9A)$$

$$\begin{cases} A+B = 1 & (i)\\ 6A+3B+C=0 & (ii) \\ 9A = 1 & (iii)\end{cases}$$

$$(iii) \rightarrow \boxed{A = 1/9}$$

$$\xrightarrow{(i)} \frac{1}{9}+B=1 \rightarrow \boxed{B=\frac{8}{9}}$$

$$\xrightarrow{(ii)} \frac{6}{9}+\frac{24}{9}+C=0$$

$$\frac{30}{9}+C=0$$

$$\boxed{C=-\frac{30}{9}=-\frac{10}{3}}$$

Step 4

Step 5

$$\int \frac{x^2+1}{x^3+6x^2+9x}\,dx = \frac{1}{9}\int\frac{1}{x}\,dx + \frac{8}{9}\int\frac{1}{x+3}\,dx - \frac{10}{3}\int\frac{1}{(x+3)^2}\,dx$$

$$= \frac{1}{9}\ln(x) + \frac{8}{9}\ln(x+3) + \frac{10}{3}\frac{1}{(x+3)} + C$$

$u=x+3$

$du=dx$

$\int\frac{1}{x+3} = \int\frac{1}{u}$

$\int x^{-2}\,dx = -x^{-1}+C$

Ex: $\displaystyle\int_{-1}^{2} \frac{1}{(x+5)(x^2+4)}\,dx$

irredu quad

$$\frac{1}{(x+5)(x^2+4)} = \frac{A}{x+5} + \frac{Bx+D}{x^2+4}$$

$\downarrow$ mult by common denom

$$0x^3+0x^2+0x+1 = (A)(x^2+4) + \underbrace{(Bx+D)(x+5)}_{Bx^2+(D+5B)x+5D}$$

$$= (\ 0\ )x^3 + (A+B)x^2 + (D+5B)x + (4A+5D)$$

$$\Rightarrow \begin{cases} A+B=0 & \text{(i)} \\ D+5B=0 & \text{(ii)} \\ 4A+5D=1 & \text{(iii)} \end{cases}$$

$\text{(i)} \rightarrow \boxed{A=-B} \xrightarrow{\text{(ii)}} \left(\frac{1+4B}{5}\right)+5B=0$

$\boxed{A=-B} \xrightarrow{\text{(iii)}} -4B+5D=1$

$\boxed{D=\frac{1+4B}{5}} \xrightarrow{\text{(ii)}}$

$\frac{4}{5}B+\frac{25}{5}B=-\frac{1}{5}$

$\frac{29B}{5}=-\frac{1}{5}$

$29B=-1$

$\boxed{B=-\frac{1}{29}}$

$\boxed{A=\frac{1}{29}}$

$D=\frac{1-{}^{4}/_{29}}{5}$

$=\frac{{}^{25}/_{29}}{5}=\frac{25}{5\cdot 29}=\frac{5}{29}$

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$$\int_{-1}^{2} \frac{1}{(x+5)(x^2+4)}\,dx = \frac{1}{29}\int_{-1}^{2} \frac{1}{x+5}\,dx + \int_{-1}^{2} \frac{-\frac{1}{29}x + 5/29}{x^2+4}\,dx$$

finish Tues