Ex: $\int \frac{1}{x^3-8}\,dx$

Step 1

$\deg(1)=0$

$\deg(x^3-8)=3$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$x^2+2x+4=0$

$x=\frac{-2\pm\sqrt{4-16}}{2}$

Step 2: $"x^3-8" = x^3-2^3 = \underbrace{(x-2)}_{\text{linear}}\underbrace{(x^2+2x+4)}_{\text{irreducible quadratic}}$

Tan 22

$x^2+bx=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$

"completing the square"

Step 3: $\frac{1}{(x-2)(x^2+2x+4)} = \frac{A_1}{x-2}+\frac{A_2x+A_3}{x^2+2x+4}$

mult by common denom

$A_2x^2+A_3x-2A_2x-2A_3$

$1 = A_1(x^2+2x+4)+(A_2x+A_3)(x-2)$

$0x^2+0x+1 = (A_1+A_2)x^2+(2A_1+A_3-2A_2)x+(4A_1-2A_3)$

$$\begin{cases} A_1+A_2 = 0 & \text{(i)} \\ 2A_1-2A_2+A_3=0 & \text{(ii)} \\ 4A_1-2A_3=1 & \text{(iii)} \end{cases}$$

From (i): $\boxed{A_1=-A_2}$ $\xRightarrow{\text{plug in to (ii)}}$ $-2A_2-2A_2+A_3=0$

$-4A_2=-A_3 \rightarrow \boxed{A_2=\frac{1}{4}A_3}$

$\rightarrow \boxed{A_3=4A_2}$

From (iii): $-4A_2-8A_2=1$

$-12A_2=1$

$\boxed{A_2=-\frac{1}{12}}$ $\boxed{A_1=\frac{1}{12}}$ $\boxed{A_3=-\frac{1}{3}}$ ← Step 4

Therefore we have

$$(\star)\ \int \frac{1}{x^3-8}\,dx = \int \frac{1}{12}\cdot\frac{1}{x-2} + \int \frac{(-\frac{1}{2})x + (-\frac{1}{3})}{x^2+2x+4}\,dx$$

$$= \frac{1}{12}\int \frac{1}{x-2}\,dx - \frac{1}{12}\int \frac{x}{x^2+2x+4}\,dx - \frac{1}{3}\int \frac{1}{x^2+2x+4}\,dx$$

$$= \frac{1}{12}\ln(x-2) - \frac{1}{12}\int \frac{x}{(x+1)^2+3}\,dx - \frac{1}{3}\int \frac{1}{(x+1)^2+3}\,dx$$

Complete square

$x^2+2x = (x+1)^2 - \left(\frac{1}{2}\right)^2$

"b=2"

$\Downarrow$

$x^2+2x+4 = (x+1)^2 - 1 + 4$

$= (x+1)^2+3$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\int \frac{x}{(x+1)^2+3}\,dx = \int \frac{u-1}{u^2+3}\,du$$

$u = x+1$

$du = dx$

$x = u-1$

$$\frac{1}{u^2+3}\left(\frac{1/3}{1/3}\right) = \frac{1}{3}\,\frac{1}{\frac{u^2}{3}+1}$$

$$= \frac{1}{3}\,\frac{1}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$$

$$= \int \frac{u}{u^2+3}\,du - \int \frac{1}{u^2+3}\,du$$

$w = u^2+3$

$dw = 2u\,du$

$\frac{1}{2}dw = u\,du$

$$= \frac{1}{2}\int \frac{1}{w}\,dw - \frac{1}{3}\int \frac{1}{\left(\frac{u}{\sqrt{3}}\right)^2+1}\,du$$

$\Omega = \frac{u}{\sqrt{3}}$

$\arctan\left(\frac{u}{\sqrt{3}}\right)$

$$= \frac{1}{2}\ln(w) - \frac{\sqrt{3}}{3}\int \frac{1}{\Omega^2+1}\,d\Omega$$

$\sqrt{3}\,d\Omega = du$

$$= \frac{1}{2}\ln(u^2+3) - \frac{1}{\sqrt{3}}\arctan(\Omega) + C$$

$$= \frac{1}{2}\ln((x+1)^2+3) - \frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

③

$$\int \frac{1}{(x+1)^2+3}\,dx \underset{\substack{u=x+1\\ du=dx}}{=} \int \frac{1}{u^2+3}\,du$$

$$= \frac{1}{3}\int \frac{1}{\left(\frac{u}{\sqrt{3}}\right)^2+1}\,du$$

$\Omega = \frac{u}{\sqrt{3}}$

$\sqrt{3}\,d\Omega = du$

$$= \frac{\sqrt{3}}{3}\int \frac{1}{\Omega^2+1}\,d\Omega$$

$$= \frac{1}{\sqrt{3}}\arctan(\Omega) + C$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

So finally,

$$\int \frac{1}{x^3-8}\,dx = \frac{1}{12}\ln(x-2) - \frac{1}{12}\left[\frac{1}{2}\ln\left((x+1)^2+3\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right)\right]$$

$$- \frac{1}{3}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

Ex: $\int \frac{1}{x^3 - x}\,dx$

Step 1 no div req

$x^2 - 1 = 0$

$x = \pm 1$

diff of sq

Step 2 $x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$

Step 3 $$\frac{1}{x(x-1)(x+1)} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{x+1}$$

$$1 = A_1\underbrace{(x-1)(x+1)}_{x^2-1} + A_2\underbrace{x(x+1)}_{x^2+x} + A_3\underbrace{x(x-1)}_{x^2-x}$$

$$0x^2 + 0x + 1 = (A_1 + A_2 + A_3)x^2 + (A_2 - A_3)x + (-A_1)$$

$$\begin{cases} A_1 + A_2 + A_3 = 0 & (i) \\ A_2 - A_3 = 0 & (ii) \\ -A_1 = 1 & (iii) \end{cases}$$

(iii) $\rightarrow \boxed{A_1 = -1}$

(i) $\rightarrow -1 + A_2 + A_3 = 0 \rightarrow \boxed{A_2 = -A_3 + 1}$

$-A_3 + 1 - A_3 = 0$

$-2A_3 = -1$

$\boxed{A_3 = \frac{1}{2}}$

$\boxed{A_2 = \frac{1}{2}}$

So,

$$\int \frac{1}{x^3-x}\,dx = \int -\frac{1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}\,dx$$

$$= -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= -\ln(x) + \frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) + C$$