Partial Fractions

Help us $\int \frac{\text{polynom}}{\text{polynom}}$

Polynomial long division

$\frac{5}{2}$ $\quad 5 \div 2$

$$\boxed{2 \ R1}$$

$$2\overline{)5}$$

bottom top

$-(4)$

1 "remainder"

Stop

$$\frac{5}{2} = 2 + \frac{1}{2}$$

$$= 2 + \frac{1}{2}$$

$(x+3)(x-1)$

$$\frac{x^2+2x-3}{x+1}$$

$$\boxed{x+1 \ R \ -4}$$

$$x+1\overline{)x^2+2x-3}$$

$-(x^2+x+0)$

$x-3$

$-(x+1)$

-4

$$= x+1 - \frac{4}{x+1}$$

$$\frac{x^2+2x+1-4}{x+1}$$

$$= 1 - \frac{4}{x+1}$$

$$\frac{x^3+2x^2-x+5}{x^2+x+1}$$

$$\begin{array}{r|l}
 & x+1 \quad R\ (-3x+4) \\
\hline
x^2+x+1 & \cancel{x^3}+2x^2-x+5 \\
 & -(\cancel{x^3}+x^2+x) \\
\hline
 & \cancel{x^2}-2x+5 \\
 & -(\cancel{x^2}+x+1) \\
\hline
 & -3x+4
\end{array}$$

$$\Rightarrow \frac{x^3+2x^2-x+5}{x^2+x+1} = x+1+\frac{-3x+4}{x^2+x+1}$$

$$= \frac{x^3+x^2+x+x^2+x+1-3x+4}{x^2+x+1}$$

$$= \frac{x^3+2x^2-x+5}{x^2+x+1} \checkmark$$

Technique of Partial Fractions to $\int \frac{P(x)}{Q(x)} dx$ polynoms

Step 1 if $\deg(P(x)) \geq \deg(Q(x))$, use polynomial long division

Step 2 factor denominator into linear $(ax+b)$ and irreducible quadratics ax^2+bx+C

u-sub

deg 2 polynoms that don't factor /// have imaginary roots

arctan

x^2+1 irreducible

$x^2+1=0$

$x=\pm\sqrt{-1}=\pm i$

x^2-1 not irreducible

$x^2-1=0$

$(x-1)(x+1)=0$

$x=1,-1$

$\int \frac{1}{x^2+1}$

$$\frac{1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

$$\frac{1}{(x+1)^2}$$

Step 3 if denominator has...

(i) $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$

THEN Guess form

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_n}{a_nx+b_n}$$

(ii) repeated linear factor $(ax+b)^n$

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$$

(iii) irreducible quadratics

$$(a_1x^2+b_1x+c_1)\cdots(a_nx^2+b_nx+c_n)$$

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}+\frac{A_2x+B_2}{a_2x^2+b_2x+c_2}+\cdots+\frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$$

(iv) repeated irreducible $(ax^2+bx+c)^n$

$$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^2}$$

Step 4) Solve for all the $\underbrace{\text{unknown constants}}_{(A_1,\ldots B_1,\ldots \text{ etc})}$

Step 5) evaluate the $\int$ using known techniques

Ex: $\displaystyle\int \frac{2x-3}{x^3+x}\,dx$

Step 1| $\deg(P(x))=1 \not\geq \deg(Q(x))=3$

no need for long div

Step 2| $x^3+x = \underset{\text{linear}}{x}\,\underbrace{(x^2+1)}_{\text{irred quad}}$

Step 3|

find these

$$\frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Step 4| $\downarrow$ multiply both sides by $x(x^2+1)$

$$0x^2 + 2x - 3 = (A)(x^2+1) + (Bx+C)x$$

$$= (A+B)x^2 + (\ C\ )x + (A)$$

"Equate coeffs"

$$\begin{cases} A + B = 0 & \text{(i)} \\ \boxed{C = 2} & \text{(ii)} \\ \boxed{A = -3} & \text{(iii)} \end{cases}$$

Plug (iii) into (i):

$$-3+B=0 \longrightarrow \boxed{B=3}$$

Step 5

$$\int \frac{2x-3}{x^3+x}\,dx \overset{\substack{\text{step 3}\\ \text{step 4}}}{=} \int \frac{-3}{x} + \frac{3x+2}{x^2+1}\,dx$$

equal!

$$= -3\int \frac{1}{x}\,dx + 3\underbrace{\int \frac{x}{x^2+1}\,dx}_{\substack{u=x^2+1\\ \frac{1}{2}du = x\,dx}} + 2\int \frac{1}{x^2+1}\,dx$$

$$= -3\ln(x) + \frac{3}{2}\underbrace{\int \frac{1}{u}\,du}_{=\ln(u)} + 2\arctan(x) + C$$

$$= -3\ln(x) + \frac{3}{2}\ln(x^2+1) + 2\arctan(x) + C$$