Integration by parts

Recall $\int f' \overset{FTC}{=} f$

the $\int$ version of product rule:

$$(fg)' = f'g + fg'$$

$$\Big\downarrow \int$$

$$\int (fg)' = \int f'g\,dt + \int fg'\,dt$$

$$\begin{array}{c} \| \\ fg \end{array}$$

More commonly written using $u = f$ $\quad v = g$

$du = f'\,dt \quad dv = g'\,dt$

$$uv = \underbrace{\int v\,du}_{} + \int u\,dv$$

$$\boxed{\int u\,dv = uv - \underbrace{\int v\,du}_{\text{compute}}}$$

start w/ integral

choose what u is

choose what dv is

Rule of thumb

* pick u so that taking its derivative "simplifies" it
* dv should be picked so its integral doesn't get more complicated

Ex: Compute $\int x \sin(x)\,dx$

Soln: Options:

$u = 1$
$dv = x\sin x\,dx$

$\downarrow$

$du = 0$
$v = \int x\sin x\,dx$

want to solve... useless!

$u = \sin(x) \quad dv = x\,dx$

$\downarrow$

$du = \cos(x)\,dx$
$v = \frac{x^2}{2}$

$\downarrow$

$$\int x\sin(x)\,dx = \frac{x^2}{2}\sin(x) - \underbrace{\frac{1}{2}\int x^2\cos(x)\,dx}_{\text{more difficult than original!}}$$

not a wise choice

$u = x \quad dv = \sin x\,dx$

$\downarrow$

$du = dx$
$v = -\cos(x)$

$\downarrow$

$$\int x\sin(x)\,dx = x(-\cos(x)) - \int(-\cos x)\,dx$$
$$= -x\cos(x) + \sin(x) + C$$

✓✓

$u = x\sin x$
$dv = 1\,dx$

$\downarrow$

$du = \sin(x) + x\cos(x)$
$v = x$

$\downarrow$

$$\int x\sin(x)\,dx = x^2\sin(x) - \int x\sin(x)\,dx - \int x^2\cos(x)\,dx$$

not a wise choice

SAME

$\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

$w = 5x$

$\frac{1}{5}dw = dx$

Ex: $\int x^2 e^{5x}\,dx$

$u = x^2 \longrightarrow du = 2x\,dx$

$dv = e^{5x}\,dx \longrightarrow v = \frac{1}{5}e^{5x}$

So, compute

(★) $$\int x^2 e^{5x}\,dx = x^2\left(\frac{1}{5}e^{5x}\right) - \int \frac{1}{5}e^{5x}(2x)\,dx$$

$$= \frac{1}{5}x^2 e^{5x} - \frac{2}{5}\boxed{\int x e^{5x}\,dx}$$

for reference later

by parts again

"n times"

Now consider $\int x e^{5x}\,dx$

$u_2 = x \longrightarrow du_2 = dx$

$dv_2 = e^{5x}\,dx \longrightarrow v_2 = \frac{1}{5}e^{5x}$

So, compute

$$\int x e^{5x}\,dx = x\left(\frac{1}{5}e^{5x}\right) - \int \left(\frac{1}{5}\right)e^{5x}\,dx$$

$$= \frac{x}{5}e^{5x} - \frac{1}{5}\int e^{5x}\,dx$$

$$= \frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} + C$$

So by (★):

$$\int x^2 e^{5x}\,dx = \boxed{\frac{1}{5}x^2 e^{5x} - \frac{2}{25}x e^{5x} + \frac{2}{125}e^{5x} + \tilde{C}} \;;\; \tilde{C} = -\frac{2}{5}C$$

Ex: Compute $\int \ln(x)\,dx$

two possibilities

$\cancel{u = 1}$

$\cancel{dv = \ln(x)dx}$

$\cancel{du = 0}$

$\cancel{v = \int \ln(x)\,dx}$

~~useless~~

$u = \ln(x) \rightarrow du = \frac{1}{x}\,dx$

$dv = 1\,dx \rightarrow v = x$

So, compute

$$\int \ln(x)dx = x\ln(x) - \int \frac{1}{x} x\,dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x + C$$

Ex: $\int \arctan(x)\,dx$

Recall

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

$u = \arctan(x) \rightarrow du = \frac{1}{1+x^2}dx$

$dv = 1\,dx \rightarrow v = x$

So compute

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

$w = 1+x^2$

$\frac{1}{2}dw = x\,dx$

$$= x\arctan(x) - \frac{1}{2}\int \frac{1}{w}\,dw$$

$$= x\arctan(x) - \frac{1}{2}\ln(w) + C$$

$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$