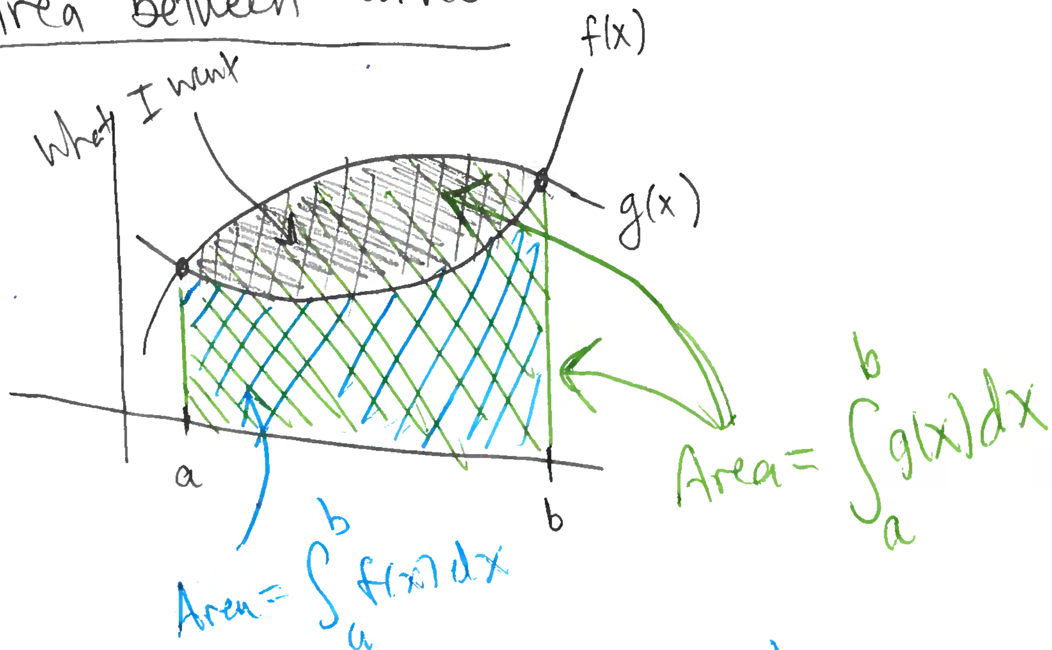
Area between curves



$$\text{Area I want} = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

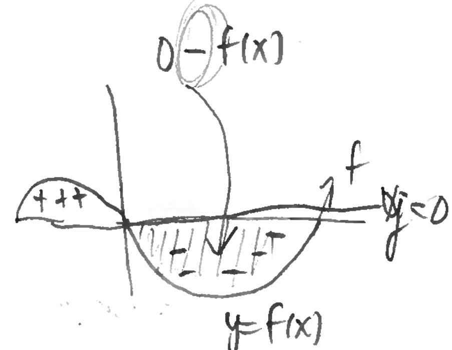
General idea

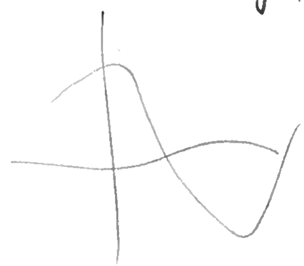
if region is bounded between g(x) on top and f(x) on bottom on [a,b]

$$\text{Area(Region)} = \int_a^b \underset{\text{top}}{g(x)} - \underset{\text{bottom}}{f(x)}\,dx$$

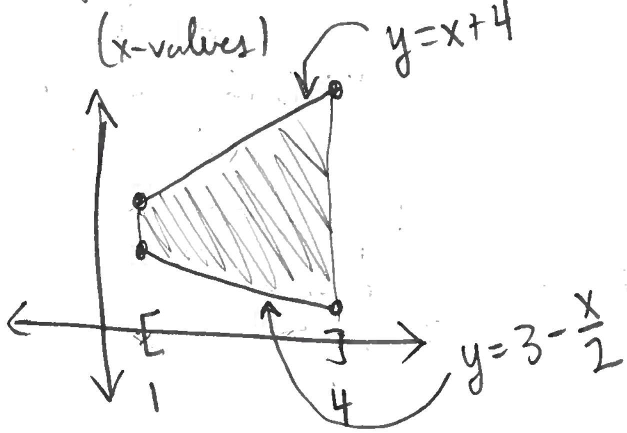
EX: Find area of region bounded by curves $y = x+4$ and $y = 3 - \frac{x}{2}$ above interval $[1,4]$.

referring to domain (x-values)





Soln:



$$\text{Area} = \int \text{top} - \text{bot}$$

$$= \int_1^4 (x+4) - \left(3 - \frac{x}{2}\right) dx$$

$$= \int_1^4 \frac{3x}{2} + 1 \, dx$$

$$= \frac{3}{2}\int_1^4 x \, dx + \int_1^4 1 \, dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_1^4 + \Big[x\Big]_1^4$$

$$= \frac{3}{2}\left[\frac{4^2}{2} - \frac{1^2}{2}\right] + \left[4-1\right]$$

$$= \frac{3}{2}\left[\frac{15}{2}\right] + 3 = \frac{45}{4} + \frac{12}{4} = \frac{57}{4}$$

$-3 - \frac{x}{2}$

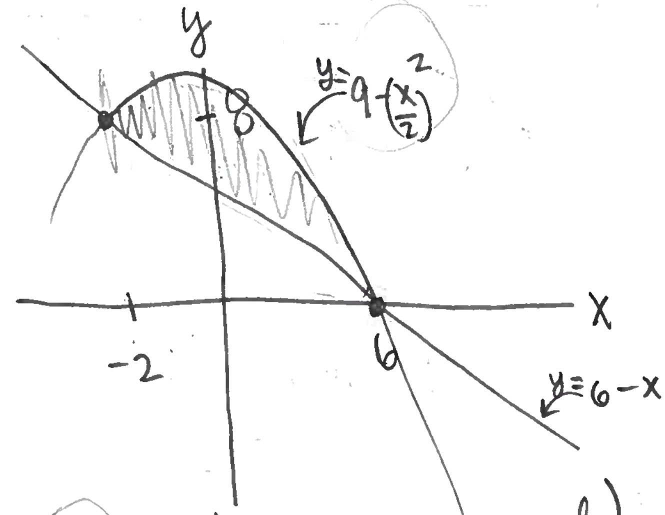
$x + \frac{x}{2} = \frac{2x}{2} + \frac{x}{2}$

$1 = x^0$

Ex: Find area of region bdd by

$y = 9 - \left(\frac{x}{2}\right)^2$ and $y = 6 - x$

Soln:



Find ∩ points (set curves equal)

$9 - \left(\frac{x}{2}\right)^2 = 6 - x$

$9 - \frac{x^2}{4} = 6 - x$

mult by 4

$36 - x^2 = 24 - 4x$

$0 = x^2 - 4x - 12$

$0 = (x-6)(x+2)$

$\boxed{x = 6, -2}$

$\left(\frac{x}{2}\right)^2 = \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)$

Area = $\int$ top − bot

$= \int_{-2}^{6} \left(9 - \left(\frac{x}{2}\right)^2\right) - (6-x)\,dx$

$= \int_{-2}^{6} -\frac{x^2}{4} + 3 + x\,dx$

$= \left[-\frac{1}{4}\frac{x^3}{3} + 3x + \frac{x^2}{2}\right]_{-2}^{6}$

$= \left(-\frac{1}{4}\cdot\frac{6^3}{3} + 3(6) + \frac{6^2}{2}\right) - \left(-\frac{1}{4}\frac{(-2)^3}{3} + 3(-2) + \frac{(-2)^2}{2}\right)$

$= \left(\frac{-216}{12} + 18 + \frac{36}{2}\right) - \left(\frac{8}{12} - 6 + \frac{4}{2}\right)$

$= (\cancel{-18} + \cancel{18} + 18) - \left(\frac{2}{3} - 6 + 2\right)$

$= 18 - \frac{2}{3} + 6 - 2 = 22 - \frac{2}{3} = \frac{64}{3}$

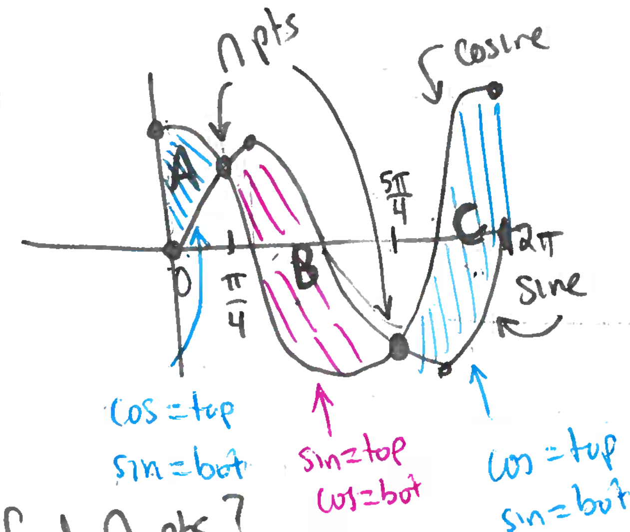
$\frac{216}{12} = \frac{108}{6} = \frac{54}{3} = 18$

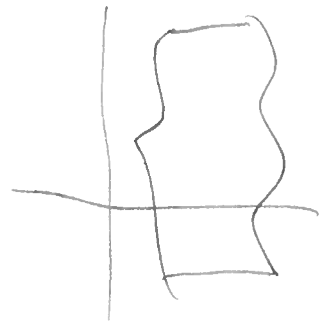
$3\overline{)54} = 18$, 3, 24

$36 \times 6 = 216$

Ex.: Find area bdd by $y=\sin(x)$ and $y=\cos(x)$ on $[0, 2\pi]$.

Soln:





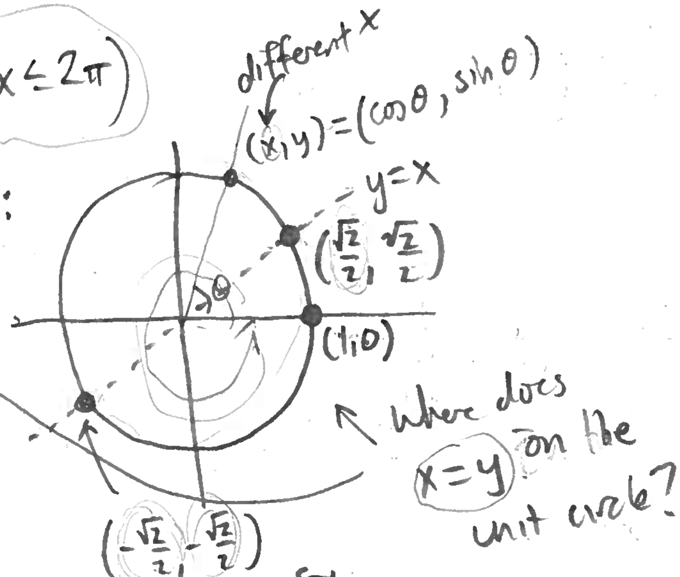
How to find ∩ pts?

$\sin(x) = \cos(x)$ $(0 \le x \le 2\pi)$

angle (b/c it's plugged into a trig function)

Recall unit circle:

$$\boxed{x = \frac{\pi}{4}, \frac{5\pi}{4}}$$



Desired Area = A + B + C

$$\text{Area}(A) = \int_0^{\pi/4} \cos(x) - \sin(x)\,dx$$

$$= \sin(x) + \cos(x)\Big|_0^{\pi/4}$$

$$= \left(\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) - \left(\sin(0) + \cos(0)\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1) = \sqrt{2} - 1$$

$$\text{Area}(B) = \int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)\,dx$$

$$= -\cos(x) - \sin(x)\Big|_{\pi/4}^{5\pi/4}$$

$$= \left(-\cos\left(\tfrac{5\pi}{4}\right) - \sin\left(\tfrac{5\pi}{4}\right)\right) - \left(-\cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)\right)$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$\text{Area}(C) = \int_{5\pi/4}^{2\pi} \cos(x) - \sin(x)\,dx$$

$$= \left[\sin(2\pi) + \cos(2\pi)\right] - \left[\sin\left(\tfrac{5\pi}{4}\right) + \cos\left(\tfrac{5\pi}{4}\right)\right]$$

$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$$= [0 + 1] + \sqrt{2}$$

Desired area $= (\sqrt{2} - \cancel{1}) + 2\sqrt{2} + (\cancel{1} + \sqrt{2}) = 4\sqrt{2}$