Completing the Square

$$\boxed{x^2+bx=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2}$$

Verify this

$$\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)\left(x+\frac{b}{2}\right)-\left(\frac{b}{2}\right)^2$$

$$=\left[x^2+\cancel{2}\left(\frac{b}{\cancel{2}}\right)x+\cancel{\left(\frac{b}{2}\right)^2}\right]-\cancel{\left(\frac{b}{2}\right)^2}$$

$$=x^2+bx$$

Ex: $\displaystyle\int\frac{1}{x^2+2x+1}\,dx=\int\frac{1}{(x+1)^2}\,dx=\int\frac{1}{u^2}\,du$

$u=x+1$

$du=dx$

$$=\int u^{-2}\,du$$

$$=\frac{u^{-1}}{-1}+C$$

$$=\frac{-1}{u}+C$$

$$=\frac{-1}{x+1}+C$$

note: $(x^2+2x)+1=(x+1)^2$

Complete square: $b=2$

$$(x^2+2x)+1=\left(x+\frac{2}{2}\right)^2-\cancel{\left(\frac{2}{2}\right)^2}+\cancel{1}$$

$$=(x+1)^2$$

Ex: $\displaystyle\int \frac{1}{x^2+2x-1}\,dx$

strange!

Complete the square:

$$(x^2+2x)-1 = (x+1)^2-1-1 = (x+1)^2-2$$

$b=2$

$$\Rightarrow \int \frac{1}{x^2+2x-1}\,dx = \int \frac{1}{(x+1)^2-2}\,dx$$

$$= \int \frac{1}{(x+1)^2-2}\,\frac{\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{1-\frac{(x+1)^2}{2}}\,dx$$

Recall

$i=\sqrt{-1}$

$\boxed{i^2=-1}$

$\boxed{e^{ix}=\cos(x)+i\sin(x)}$

$u=\dfrac{x+1}{\sqrt{2}\,i}$

$du=\dfrac{1}{\sqrt{2}\,i}\,dx$

$\sqrt{2}\,i\,du=dx$

$$= -\frac{1}{2}\int \frac{1}{1+\left(\frac{x+1}{\sqrt{2}\,i}\right)^2}\,dx$$

$$= -\frac{\sqrt{2}\,i}{2}\int \frac{1}{1+u^2}\,du$$

$$= -\frac{i}{\sqrt{2}}\arctan(u)+C$$

$$= -\frac{i}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}\,i}\right)+C$$

Ex: $\int \frac{1}{x^2+10x+27}\,dx$

$\int \frac{1}{1+x^2}\,dx = \arctan(x)+C$

Complete square:

$$(x^2+10x)+27 = [(x+5)^2-5^2]+27$$

$b=10$

$$= (x+5)^2+2$$

$$\Rightarrow \int \frac{1}{x^2+10x+27}\,dx = \int \frac{1}{(x+5)^2+2}\,dx$$

$$= \int \frac{1}{(x+5)^2+2}\cdot\frac{1/2}{1/2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\frac{(x+5)^2}{2}+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{x+5}{\sqrt{2}}\right)^2+1}\,dx$$

$u = \frac{x+5}{\sqrt{2}}$

$du = \frac{1}{\sqrt{2}}\,dx$

$\sqrt{2}\,du = dx$

$$= \frac{\sqrt{2}}{2}\int \frac{1}{u^2+1}\,du$$

$$= \frac{\sqrt{2}}{2}\arctan(u)+C$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x+5}{\sqrt{2}}\right)+C$$

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$$\int \frac{1}{\sqrt{-x^2-14x-44}}\,dx$$

$b=14$

$$-x^2-14x-44 = -(x^2+14x+44)$$

$$= -\left((x+7)^2-7^2+44\right)$$

$$= -\left((x+7)^2-5\right)$$

$$= 5-(x+7)^2$$

$$1-\frac{(x+7)^2}{5}$$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-14x-44}}\,dx = \int \frac{1}{\sqrt{5-(x+7)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{5-(x+7)^2}}\left(\frac{1/\sqrt{5}}{1/\sqrt{5}}\right)dx$$

$u = \frac{x+7}{\sqrt{5}}$

$\sqrt{5}\,du = dx$

$$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-\left(\frac{x+7}{\sqrt{5}}\right)^2}}\,dx$$

$=1$

$$= \frac{\sqrt{5}}{\sqrt{5}}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \arcsin(u) + C$$

$$= \arcsin\left(\frac{x+7}{\sqrt{5}}\right) + C$$

<u>Ex</u>: $\int \frac{1}{\sqrt{x^2+4x+4}}\,dx$

complete square: $\overset{b=4}{(x^2+4x)}+4 = [(x+2)^2 - \cancel{2^2}] + \cancel{4}$

$$\int \frac{1}{\sqrt{x^2+4x+4}}\,dx = \int \frac{1}{\sqrt{(x+2)^{\cancel{2}}}}\,dx$$

$u = x+2$

$du = dx$

$$= \int \frac{1}{x+2}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln(u) + C$$

$$= \ln(x+2) + C$$

Ex:

$$\int \frac{\arctan(x)}{1+x^2}\,dx$$

$u = \arctan x$

$du = \frac{1}{1+x^2}\,dx$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\arctan(x))^2}{2} + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x)$$

$$\Big\updownarrow \frac{d}{dx}$$

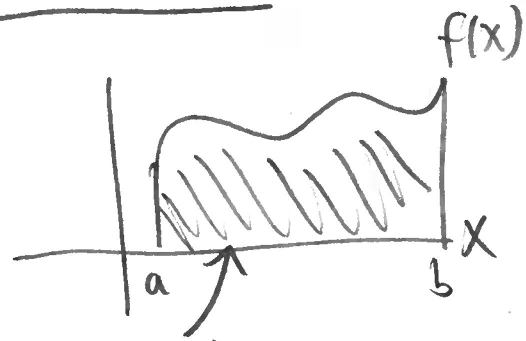
$$\frac{1}{x^2+1} = \frac{d}{dx}\arctan(x)$$

$$\cancel{\tan^{-1}(x) \neq \frac{1}{\tan(x)}}$$

$$\tan^{-1}(x) = \arctan(x)$$

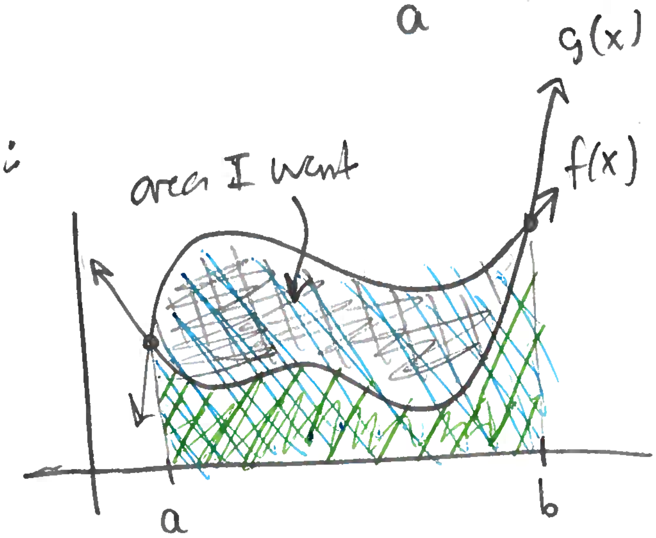
Area between curves

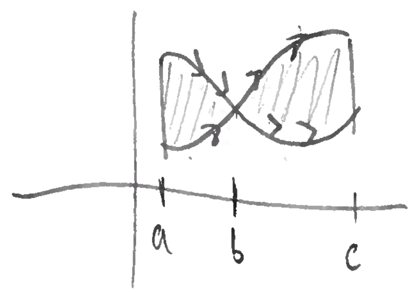
You know:



area $= \int_a^b f(x)\,dx$

Now:





$\int_a^b f(x)$

$\int_a^b g(x)\,dx$

area I want $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$= \int_a^b f(x) - g(x)\,dx$

$\int$ top fnct - bottom fnct = area between