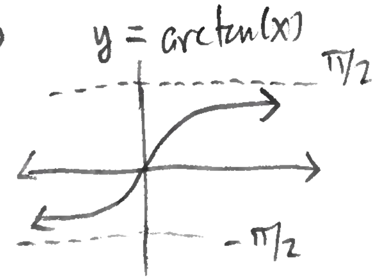
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \qquad -\infty < x < \infty$$



$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \leftarrow -1 \le x \le 1$$

$$-\int \frac{1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C \leftarrow -1 \le x \le 1$$

Ex: $\displaystyle\int_0^{1/4} \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_0^{1/4}$

$$= \arctan\left(\tfrac{1}{4}\right) - \overset{0}{\arctan(0)}$$

Ex: $\displaystyle\int \frac{1}{2+x^2}\,dx = \int \frac{1}{(2+x^2)}\overbrace{\left(\frac{1/2}{1/2}\right)}^{=1}dx$

looks a LOT like arctan but a little different (the "2")

Need to =1

$$= \frac{1}{2}\int \frac{1}{1+\frac{x^2}{2}}\,dx \qquad \frac{x^2}{2} = \left(\frac{x}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2}\int \frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}\,dx = \frac{1}{2}\int \frac{\sqrt{2}}{1+u^2}\,du$$

$u = \frac{x}{\sqrt{2}}\,dx$

$du = \frac{1}{\sqrt{2}}\,dx$

$\sqrt{2}\,du = dx$

$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{2}}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{\sqrt{2}}\arctan(u) + C$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

Ex:
$$\int \frac{1}{\sqrt{7-x^2}}\,dx$$

Compare to
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

2

$\cancel{\sqrt{49}=7}$

$\sqrt{a}\sqrt{b} = \sqrt{ab}$ (if $a,b>0$)

$1 = \sqrt{1} = \sqrt{-1(-1)} = \sqrt{-1}\sqrt{-1} = -1$ ← breaks

$$= \int \frac{1}{\sqrt{7-x^2}}\left(\frac{\sqrt{1/7}}{\sqrt{1/7}}\right)dx$$

$$= \sqrt{1/7}\int \frac{1}{\sqrt{1-\frac{x^2}{7}}}\,dx$$

$$= \sqrt{\frac{1}{7}}\int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{7}}\right)^2}}\,dx$$

$u = \frac{x}{\sqrt{7}}$

$\sqrt{7}\,du = dx$

$$= \frac{1}{\cancel{\sqrt{7}}}\int \frac{\cancel{\sqrt{7}}}{\sqrt{1-u^2}}\,du$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \arcsin(u) + C$$

$$= \arcsin\left(\frac{x}{\sqrt{7}}\right) + C$$

ss5

777711

7

$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$ ③

Ex: $\int \frac{1}{1+5x^2}\,dx$

$= \int \frac{1}{1+(\sqrt{5}x)^2}\,dx$ $\quad u = \sqrt{5}\,x$

$\frac{1}{\sqrt{5}}\,du = dx$

$= \frac{1}{\sqrt{5}} \int \frac{1}{1+u^2}\,du$

$= \frac{1}{\sqrt{5}} \arctan(u) + C$

$= \frac{1}{\sqrt{5}} \arctan(\sqrt{5}\,x) + C$

Ex: $\int \frac{1}{9+11x^2}\,dx = \int \frac{1}{9+11x^2}\left(\frac{1/9}{1/9}\right)dx$

$= \frac{1}{9} \int \frac{1}{1+\frac{11}{9}x^2}\,dx$

$u = \sqrt{\frac{11}{9}}\,x$

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ $(a,b>0)$

$\sqrt{\frac{9}{11}}\,u = dx$

$= \frac{1}{9} \int \frac{1}{1+\left(\sqrt{\frac{11}{9}}\,x\right)^2}\,dx$

$= \frac{1}{9} \int \frac{\frac{3}{\sqrt{11}}}{1+u^2}\,du = \frac{1}{3\sqrt{11}} \arctan(u) + C$

$= \frac{1}{3\sqrt{11}} \arctan\left(\frac{\sqrt{11}}{3}x\right) + C$

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Ex: $\displaystyle\int \frac{x+8}{5+x^2}\,dx$

arctan

$u = 5+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \int \frac{x}{5+x^2}\,dx + \int \frac{8}{5+x^2}\,dx$$

$w = \frac{x}{\sqrt{5}}$

$\sqrt{5}\,dw = dx$

$$= \frac{1}{2}\int \frac{1}{u}\,du + \frac{8}{5}\int \frac{1}{1+\left(\frac{x}{\sqrt{5}}\right)^2}\,dx$$

$$= \frac{1}{2}\ln(u) + C + \frac{8\sqrt{5}}{5}\int \frac{1}{1+w^2}\,dw$$

$$= \frac{1}{2}\ln(5+x^2) + C + \frac{8}{\sqrt{5}}\arctan(w) + D$$

$$= \frac{1}{2}\ln(5+x^2) + \frac{8}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + \tilde{C}, \quad \tilde{C} = C+D$$

Completing the square

Ex: $x^2+2x+1 = (x+1)^2$ factored

$x^2+2x+2 = (x^2+2x+2+1)-1$

$= (x+1)^2+1$

$$\left(x+\frac{b}{2a}\right)^2$$

$$= \left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right)$$

$$= x^2+\frac{b}{2a}x+\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2$$

in general

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$= a\left(x^2+\frac{b}{a}x+\frac{c}{a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2+\underbrace{\left(\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right)}_{\#}\right)$$