Theorem 25: Suppose that each of f and g are functions that are differentiable at the point p and that h is the function defined by $h(x) = f(x) + g(x)$ for all $x \in D_f$. Show that h is also differentiable at the point p.

Proof. Let D_f be the derivative of f at p, and let D_g be the derivative of g at p. Let $\epsilon > 0$. By definition, $\exists \delta_f$ such that if $\left|\frac{f(t)-f(p)}{t-p} - D_f\right| < \frac{\epsilon}{2}$. Similarly, $\exists \delta_g$ such that if $\left|\frac{g(t)-g(p)}{t-p} - D_g\right| < \frac{\epsilon}{2}$.
We know that p is a limit point of $dom(f)$ and p is a limit point of $dom(g)$. Suppose p is not a limit point of $dom(h) = dom(f) \cap dom(g)$. Then there exists an open interval S containing p such that $S \cap dom(h) = \{p\}$. This is impossible because $(p - \delta_f, p + \delta_f) \cap (p - \delta_g, p + \delta_g) \subset dom(h)$.
Pick $\delta = min\{\delta_f, \delta_g\}$. Let $D = D_f + D_g$. If $|t - p| < \delta$, then calculate

$$\begin{aligned}
\left|\frac{h(t)-h(p)}{t-p} - D\right| &= \left|\frac{f(t)+g(t)-(f(p)+g(p))}{t-p} - (D_f + D_g)\right| \\
&= \left|\left[\frac{f(t)-f(p)}{t-p} - D_f\right] + \left[\frac{g(t)-g(p)}{t-p} - D_g\right]\right| \\
&\leq \left|\frac{f(t)-f(p)}{t-p} - D_f\right| + \left|\frac{g(t)-g(p)}{t-p} - D_g\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Thus, h is differentiable at p. □