Math 4590

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Theorem 16

If $I_1, I_2, I_3, ...$ is a sequence of closed intervals so that for each positive integer n, $I_{n+1} \subseteq I_n$, and the length of I_n is less than $\frac{1}{n}$, then there is only one point p such that for each positive integer n, $p \in I_n$.

Proof.

Let $I_1, I_2, I_3, ...$ be a sequence of closed intervals such that $I_{n+1} \subseteq I_n$ and $len(I_n) < \frac{1}{n}$.
Also, let $I_n = [a_n, b_n]$.
Now we call the point set $M_L = \{a_1, a_2, a_3, ...\}$, which is bounded above by b_1.
By the Completeness Axiom, there is either a right-most point of M_L or a first point to the right of M_L.
So by Theorem 10, the sequence $\{a_n\}$ converges to some point A.
Similarly, $M_R = \{b_1, b_2, b_3, ...\}$ is bounded below by a_1 and the sequence $\{b_n\}$ converges to some point B.
Now, by contradiction, suppose $A \neq B$.
Then $d(A, B) > 0$.
Since $\{\frac{1}{n}\}$ converges to 0, then $\exists N \in \mathbb{Z}^+$ such that $\forall n \in \mathbb{Z}^+$ with $n \geq N$, $\frac{1}{n} \in (A, B)$.
And as $len(I_n) = |a_n - b_n| < \frac{1}{n}$, then $|a_n - bn| < |A - B|$.
Thus, $\forall n \geq N$, $I_n \subseteq [A, B]$.
Then,

$$0 \leq len(I_n) < \frac{1}{n}$$

$$0 \leq len([a_n, b_n]) < \frac{1}{n}$$

$$0 \leq |a_n - b_n| < \frac{1}{n}$$

Now let $c_n = |a_n - b_n|$.
Let S be an open interval containing 0.
Since $\{\frac{1}{n}\}$ converges to 0, then $\exists N \in \mathbb{Z}^+$ such that if $n \in \mathbb{Z}^+$ and $n \geq N$, then $|\frac{1}{n} - 0| \in S$.
Since $len(I_n) = |a_n - b_n| = c_n < \frac{1}{n}$, then $c_n \in S$.
So $\{c_n\} = \{|a_n - b_n|\}$ converges to 0.
Hence, $A = B$.
$\therefore$ If $I_1, I_2, I_3, ...$ is a sequence of closed intervals so that for each positive integer n, $I_{n+1} \subseteq I_n$, and the length of I_n is less than $\frac{1}{n}$, then there is only one point A such that for each positive integer n, $A \in I_n$. ■