Math 4590

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Theorem 13:
If each of f and g is a a function, x is a point in the domain of each of f and g, f is continuous at the point $(x, f(x))$, g is continuous at the point $(x, g(x))$, and $h = f + g$, then h is continuous at the point $(x, h(x))$.

Proof.
Since f is continuous at $(x, f(x))$, then $\forall \varepsilon > 0$, $\exists \delta_f > 0$ such that if $t \in dom(f)$ and $|x - t| < \delta_f$ then $|f(x) - f(t)| < \varepsilon$.

Similarly, as g is continuous at $(x, g(x))$, then $\forall \varepsilon > 0$, $\exists \delta_g > 0$ such that if $t \in dom(g)$ and $|x - t| < \delta_g$ then $|g(x) - g(t)| < \varepsilon$.

Let $\varepsilon > 0$.

So $\exists \delta_f, \delta_g$ such that if $t \in dom(f)$, $t \in dom(g)$, $|x - t| < \delta_f$, and $|x - t| < \delta_g$, then $|f(x) - f(t)| < \frac{\varepsilon}{2}$ and $|g(x) - g(t)| < \frac{\varepsilon}{2}$.

Hence, if $t \in dom(h)$ and $|x - t| < \delta$, where $\delta = min\{\delta_f, \delta_g\}$, then:

$$\begin{aligned} |h(x) - h(t)| &= |(f(x) + g(x)) - (f(t) + g(t))| \\ &= |(f(x) - f(t)) + (g(x) - g(t))| \\ &\leq |f(x) - f(t)| + |g(x) - g(t)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, $|h(x) - h(t)| < \varepsilon$.

$\therefore$ If each of f and g is a a function, x is a point in the domain of each of f and g, f is continuous at the point $(x, f(x))$, g is continuous at the point $(x, g(x))$, and $h = f + g$, then h is continuous at the point $(x, h(x))$. ■