Theorem 11

If M has p as a limit point, then there exists either an increasing or decreasing sequence of points M converging to p.

Proof: Let M be a point set with limit point p. Let (a, b) be an open interval containing p. Thus, $\exists m_1 \in M \cap (a, b), m_1 \neq p$. Let $\in_n = min\left\{|p - m_n|, \frac{1}{2^n}\right\}$. So $\in_1 = min\left\{|p - m_1|, \frac{1}{2}\right\}$. Now let $(p - \in_1, p + \in_1)$ be an open interval containing p. Again, $\exists m_2 \in M \cap (p - \in_1, p + \in_1), m_2 \neq p$ and $\in_2 = min\left\{|p - m_2|, \frac{1}{2^2}\right\}$. Repeating this action provides us with an infinite set M, and thus either $\{m_i : m_i \in M, m_1 < p\}$, $\{m_i : m_i \in M, m_i > p\}$, or both are infinite. Without loss of generality, consider $\{m_i : m_i \in M, m_1 < p\}$ to be infinite and order these in such a way that $m_{n_1} < m_{n_2} < m_{n_3} < \cdots < p$. Pick $\in > 0$ and $N \in \mathbb{Z}^+, N = \frac{\ln(\frac{1}{\in})}{\ln(2)}$ and $n \in \mathbb{Z}^+, n > N$. Thus,

$$n > \frac{\ln(\frac{1}{\in})}{\ln(2)}$$

$$n \ln(2) > \ln(\frac{1}{\in})$$

$$\ln(2^n) > \ln(\frac{1}{\in})$$

$$e^{\ln(2^n)} > e^{\ln(\frac{1}{\in})}$$

$$2^n > \frac{1}{\in}$$

$$\in > \frac{1}{2^n}$$

Thus, $\left|p - m_{n_i}\right| < \in$ and the increasing sequence converges to p. The proof when $\{m_i : m_i \in M, m_i > p\}$ is infinite works similarly. Thus, there exists a decreasing sequence converging to p. ■