Problem 12

For each positive integer n, let $P_{2n-1} = {}^1/_{2n-1}$ and $P_{2n} = 1 + {}^1/_{2n}$

Q: Does sequence $p_1, p_2, \ldots$ converge to 0?

Solution:

$P_1 = {}^1/_1 = 1$

$P_2 = 1 + {}^1/_2$

$P_3 = {}^1/_3$

$P_4 = 1 + {}^1/_4$ … And so on …

Claim 1: The sequence $q_n = {}^1/_{2n-1}$ converges to 0.

Proof: Let $0 \in (a, b)$ be a n open interval. Choose $\epsilon = \min\{ |a|, |b|\}$ so $(-\epsilon, \epsilon) \subset (a, b)$. Pick $N \in \mathbb{Z}^+$ such that $N > {}^1/_{2\epsilon} + {}^1/_2$. If $n > N$, then $n > {}^1/_{2\epsilon} + {}^1/_2$. By axioms, $2n > {}^1/_\epsilon + 1$

$$2n - 1 > {}^1/_\epsilon$$

$${}^1/_{2n-1} < \epsilon$$

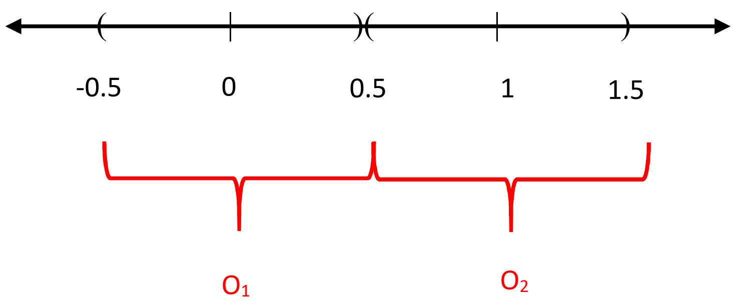
So, $|0 - {}^1/_{2n-1}| = {}^1/_{2n-1} < \epsilon$.

Claim 2: The sequence $r_n = 1 + {}^1/_{2n}$ converges to 1.

Proof: Let $1 \in (a, b)$ be an open interval. Choose $\epsilon = \min\{ |-a|, |-b|\}$. Pick $N \in \mathbb{Z}^+$ such that $N > {}^1/_{2\epsilon}$. If $n > N$, then $n > {}^1/_{2\epsilon}$, so ${}^1/_n < 2\epsilon$ and ${}^1/_{2n} < \epsilon$.

So, $|1 - r_n| = |1 - (1 + {}^1/_{2n})| = {}^1/_{2n} < \epsilon$.

Claim 3: The sequence $p_1, \ldots$ does not converge to 0.



By claim 1, there is an N_1 such that for all odd $n > N_1$, $P_n \in O_1$.

By claim 2, there is an N_2 such that for all even $n > N_2$, $P_n \in O_2$.

Since $O_1 \cap O_2 = \emptyset$, the sequence does not converge to 0.