Theorem: If $x, y \in \mathbb{R}$, then $|x+y| \leq |x| + |y|$.
Proof:

Case 1: $x \geq 0$ and $y \geq 0$

Proof of case 1: In this case, $|x+y| = x+y \leq x+y = |x|+|y|$.

Case 2: $x \leq 0$ and $y \leq 0$

Proof of case 2: In this case, $|x| = -x$ and $|y| = -y$. So,

$$|x+y| = |-\underbrace{(-x-y)}_{\geq 0}| = -x-y = |x|+|y|.$$

Case 3: $x \geq 0$ and $y \leq 0$.

Proof of case 3: In this case, $|x| = x$ and $|y| = -y$.

SubCase 1: $|x| \leq |y|$

Proof of subcase 1: This means that $x \leq -y$. Adding y gives $x+y \leq 0$. Since x is positive, $y \leq x+y \leq 0$. We can conclude that $0 \leq -(x+y) \leq -y$. Therefore, $|x+y| = -(x+y) \leq -y = |y| \leq |x|+|y|$.

SubCase 2: $|x| \geq |y|$

Proof of subcase 2: In this case, $x \geq -y$. Adding y yields $x+y \geq 0$. Since y is negative, $x \geq x+y \geq 0$. Therefore,

$$|x+y| = x+y \leq x = |x| \leq |x|+|y|.$$

Case 4: $x \leq 0$ and $y \geq 0$

Proof of case 4: The same as Case 3. Just interchange "x" and "y" in that proof and that will do it.

Since we have exhausted all possible cases of inequalities between x and y, we have completed the proof. ■