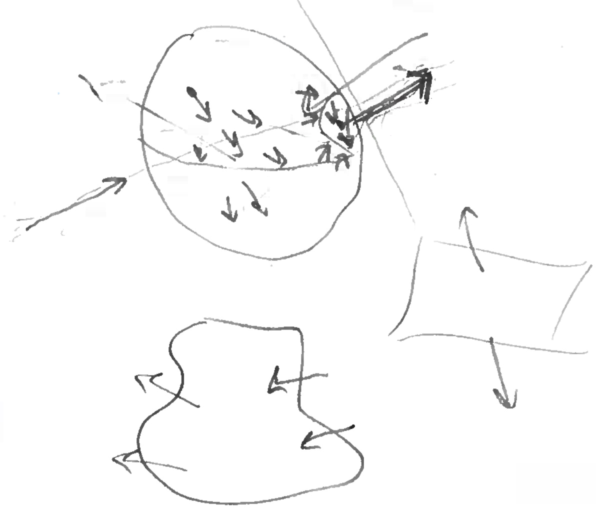
Def: If $\vec{F}$ is a vector field on an oriented surface S parametrized by $\vec{r}(u,v)$, $(u,v)\in D$ then the flux of $\vec{F}$ over S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)$$

(or $\vec{r}_v \times \vec{r}_u$)



Ex: Find flux of $\vec{F} = \langle z, y, x\rangle$ across unit sphere $x^2+y^2+z^2=1$.

Soln: Parametrize sphere w/ spherical coords

$$\vec{r}(\phi,\theta) = \langle s(\phi)c(\theta),\ s(\phi)s(\theta),\ c(\phi)\rangle$$

$0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$$r_\phi \times r_\theta = \langle s^2(\phi)c(\theta),\ s^2(\phi)s(\theta),\ s(\phi)c(\phi)\rangle$$

$\phi = \frac{\pi}{4}, \theta = \frac{\pi}{4}$

$\vec{r} = \langle \frac{2}{4}, \frac{2}{4}, \frac{\sqrt{2}}{2}\rangle$

$= \langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\rangle$

$r_\phi \times r_\theta = \langle \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{1}{2}\rangle$

$$\vec{F}(\vec{r}(\phi,\theta)) = \langle c(\phi),\ s(\phi)s(\theta),\ s(\phi)c(\theta)\rangle$$

$$r_\phi \times r_\theta = \langle s^2(\phi)c(\theta),\ s^2(\phi)s(\theta),\ s(\phi)c(\phi)\rangle$$

$$\vec{F}(\vec{r})\cdot(\vec{r}_\phi \times \vec{r}_\theta) = s^2(\phi)c(\theta)c(\phi) + s^3(\phi)s^2(\theta) + s^2(\phi)c(\theta)c(\phi)$$

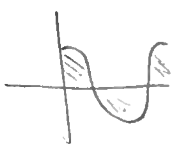
$$= s^2(\phi)\left[\underline{c(\theta)c(\phi)} + s(\phi)s^2(\theta) + \underline{c(\theta)c(\phi)}\right]$$

same

$$= s^2(\phi)\left[2c(\theta)c(\phi) + s(\phi)s^2(\theta)\right]$$

So,

$$\iint_S \vec{F}\cdot d\vec{r} = \int_0^{\pi}\int_0^{2\pi} s^2(\phi)\left[2c(\theta)c(\phi) + s(\phi)s^2(\theta)\right]d\theta\, d\phi$$



$$= \int_0^{\pi} s^2(\phi)c(\phi)\,d\phi \int_0^{2\pi} 2c(\theta)\,d\theta + \int_0^{\pi} s^3(\phi)\,d\phi \int_0^{2\pi} s^2(\theta)\,d\theta$$

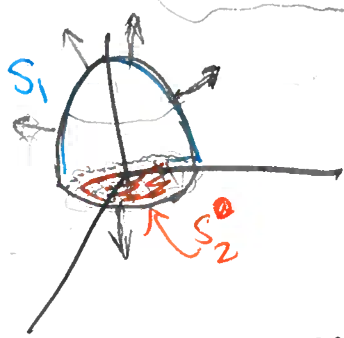
$=0$ $\qquad$ $s^2 = 1 - c^2$ $\qquad$ reduct.

$$= \dots = \frac{4\pi}{3}$$

Ex: Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, $\vec{F}=\langle y, x, z\rangle$

+ S is bdy of solid region E enclosed by paraboloid $z=1-x^2-y^2$ + plane $z=0$.

Soln:



S_1

$\vec{r}(u,v)=\langle u, v, 1-u^2-v^2\rangle$

$r_u=\langle 1, 0, -2u\rangle$

$r_v=\langle 0, 1, -2v\rangle$

$(u,v)\in$  $D=\{(r,\theta): 0\le\theta\le 2\pi,\ 0\le r\le 1\}$

$r_u\times r_v=\langle 2u, -(-2v), 1\rangle$

$F(r(u,v))=\langle v, u, 1-u^2-v^2\rangle$

$\vec{F}(\vec{r})\cdot(r_u\times r_v)=2uv+2uv+1-u^2-v^2$

$u=r\cos\theta$
$v=r\sin\theta$
$u^2+v^2=r^2$

Calculate

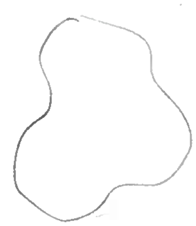
$$\iint_S \vec{F}\cdot d\vec{S}=\iint_D 4uv+1-(u^2+v^2)\,dA$$

$$=\int_0^{2\pi}\int_0^1\left(4r^2\cos(\theta)\sin(\theta)+1-r^2\right)r\,dr\,d\theta$$

(extra r)

$$=\dots$$

$$=\frac{\pi}{2}$$



(S_2) $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0 \rangle$

$$\vec{r}_r = \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle 0, 0, r\cos^2(\theta) - (-r\sin^2(\theta)) \rangle$$

$= \langle 0, 0, r \rangle$ ← points in ⊕ z dir!!

So we actually need

$$r_\theta \times r_r = \langle 0, 0, -r \rangle$$

$$\vec{F}(\vec{r}(r,\theta)) = \langle r\sin\theta, r\cos\theta, 0 \rangle$$

$$\vec{F}(\vec{r}) \cdot (\vec{r}_\theta \times \vec{r}_r) = 0 + 0 + 0 = 0$$

$$\Rightarrow \iint_D \vec{F}(\vec{r}(r,\theta)) \cdot (\vec{r}_\theta \times \vec{r}_r)\, dA = 0$$

$= \iint_S \vec{F} \cdot d\vec{S}$