Ex: Find surface area of paraboloid

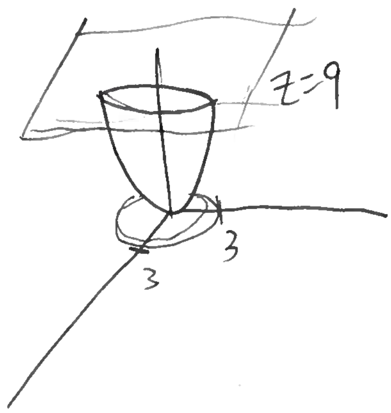
$$z = x^2+y^2 \leftarrow f(x,y)$$

lying under plane $z=9$.

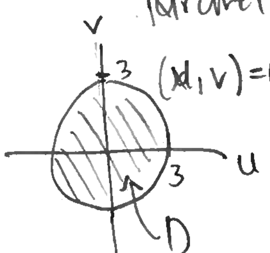
recall

$y = f(x) \rightsquigarrow \vec{r}(t) = \langle t, f(t)\rangle$

Soln:



Parametrize surface:



$(u,v) = (x,y) \in$ interior of circle of Rad 3

$$\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$$

$$(u,v) \in \{\text{circle of rad } 3\} = \left\{(r,\theta) : \begin{matrix} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{matrix}\right\}$$

$$r_u = \langle 1, 0, 2u\rangle$$

$$r_v = \langle 0, 1, 2v\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -2u, -(2v), 1\rangle$$

$$\|r_u \times r_v\| = \sqrt{1+4u^2+4v^2} = \sqrt{1+4(u^2+v^2)}$$

extra

$$\text{Surface area} = \iint_D \|r_u \times r_v\|\, dA = \int_0^{2\pi}\int_0^3 \left(\sqrt{1+4r^2}\right) r\, dr\, d\theta$$

$u = r\cos\theta$

$v = r\sin\theta$

$\Downarrow$

$u^2+v^2 = r^2$

$u = 1+4r^2$

$\frac{du}{8} = r\,dr$

$$= \int_0^{2\pi}\int_1^{37} \frac{1}{8} u^{1/2}\, du\, d\theta = \ldots$$

$$= \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

Recall: $\text{Area}(D) = \iint_D 1\, dA$ (*)

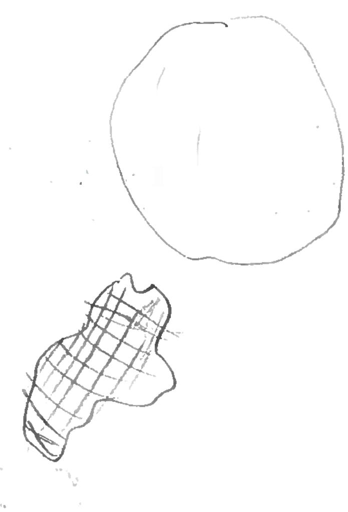
$\text{Vol}(B) = \iiint_B 1\, dV$

Def: (Surface integral) Spz S smooth surface parametrized by $r(u,v)$, $(u,v) \in D$. Then (scaler) surface integral of f over S is

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, \|\vec{r_u} \times \vec{r_v}\|\, dA$$

Note: Similar to (*):

$$\text{Surf Area} = \iint_S 1\, dS$$



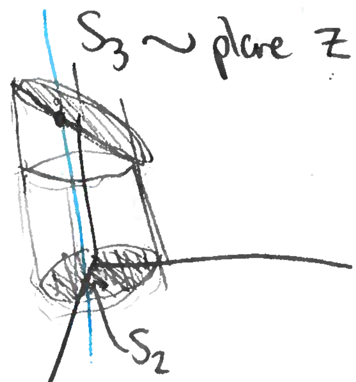
Ex: Find $\iint_S z\, dS$ where S is surface

w/ sides

$S_1 \sim$ cylinder $x^2+y^2=1$

$S_2 \sim$ disk $x^2+y^2 \le 1$ (in plane)

$S_3 \sim$ plane $z = 1+x$ lying above S_2



$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$\boxed{\text{For } S_1}$ Use cylindrical coords:

$$\begin{cases} \vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle \\ 0 \le \theta \le 2\pi, \quad 0 \le z \le 1+x = 1+\cos(\theta) \end{cases}$$

$$\begin{array}{cccc} & i & j & k \\ r_\theta = & \langle -s(\theta), & c(\theta), & 0 \rangle \\ r_z = & \langle 0, & 0, & 1 \rangle \end{array}$$

$$r_\theta \times r_z = \langle \cos(\theta), -(-\sin(\theta)), 0 \rangle$$

$\|r_\theta \times r_z\| = 1$. So, $\displaystyle\iint_{S_1} z\, dS = \iint_D z \overbrace{\|r_\theta \times r_z\|}^{=1} dS$

$$= \int_0^{2\pi} \int_0^{1+\cos(\theta)} z\, dz\, d\theta$$

$$= \dots = \frac{3\pi}{2}$$

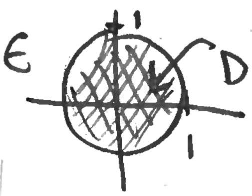
$\boxed{S_2}$ Not much to do ~ just notice on this surface $z=0$.

So,

$$\iint_{S_2} z\, dS = \iint_{S_2} 0\, dS = 0$$

$\boxed{S_3}$ Since $z = 1+x$:

$$\vec{r}(u,v) = \langle u, v, 1+u \rangle$$

$(u,v) \in$ 

$$r_u = \langle 1, 0, 1 \rangle$$

$$r_v = \langle 0, 1, 0 \rangle$$

$$r_u \times r_v = \langle -1, -0, 1 \rangle \rightarrow \|r_u \times r_v\| = \sqrt{2}$$

So,

$$\iint_{S_3} z\, dS = \iint_D (1+u)\sqrt{2}\, dA$$

$u = r\cos\theta$
$v = r\sin\theta$

$$= \int_0^{2\pi}\int_0^1 (1 + r\cos(\theta))\sqrt{2}\, \overset{\text{extra}}{r}\, dr\, d\theta$$

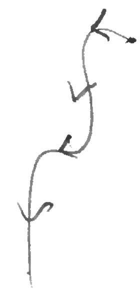
$$= \ldots = \sqrt{2}\,\pi$$

Therefore,

$$\iint_S z\, dS = \frac{3\pi}{2} + 0 + \pi\sqrt{2} = \left(\frac{3}{2} + \sqrt{2}\right)\pi$$

Oriented surfaces

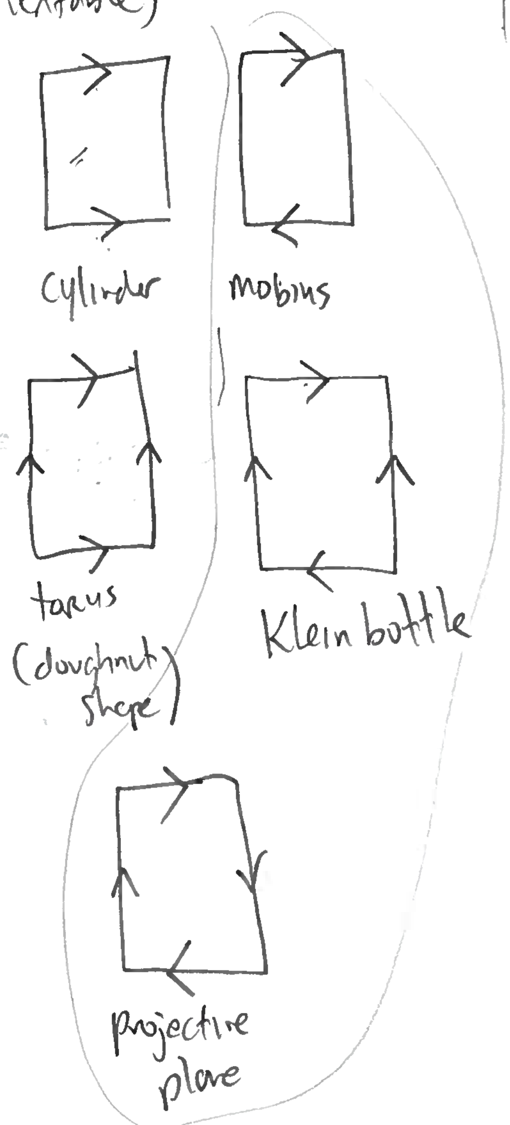
A surface is called orientable if it is possible to choose $\vec{n}$ so it varies continuously over surface.



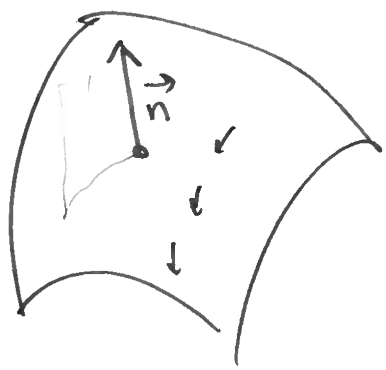
Ex: Möbius strip (not orientable)

<see video>

Klein bottle



For calculus, we need oriented surfaces.



Choosing $\vec{n}$ or $-\vec{n}$ decides the orientation of the surface you are using.