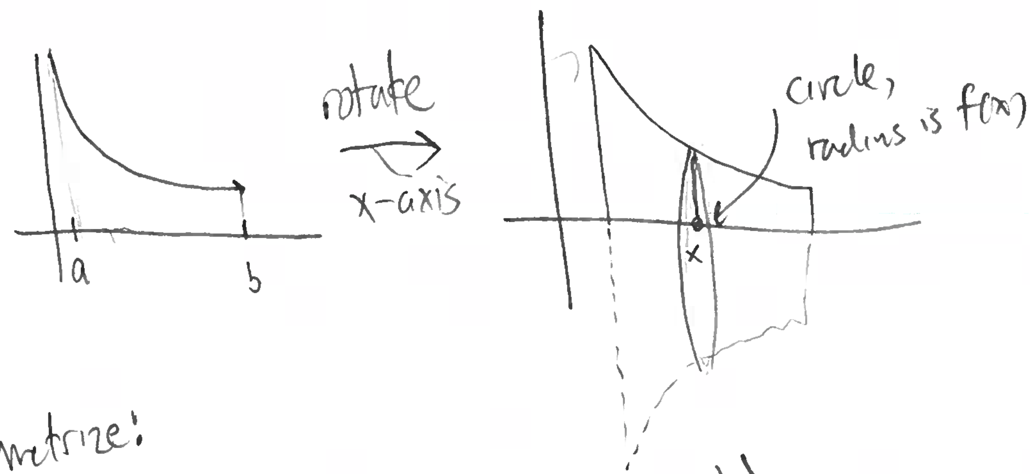
Surfaces of revolution

Recall: rotate $f(x)$ about an axis



Parametrize:

$$\vec{r}(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta) \rangle$$

$$0 \le \theta \le 2\pi$$

$$a \le x \le b$$

Ex: Rotate $f(x) = 1 - x^2$ about x-axis
$0 \le x \le 1$

$$\begin{cases} \vec{r}(x,\theta) = \langle x, (1-x^2)\cos(\theta), (1-x^2)\sin(\theta) \rangle \\ 0 \le x \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$

EX: Find eqt of tangent plane of surface

$\vec{r}(u,v) = \langle \overset{x}{u}, \overset{y}{2v^2}, \overset{z}{u^2+v} \rangle$

at $(2,2,3)$

Which u,v give us $(\overset{}{\underset{x}{2}}, \underset{y}{2}, \underset{z}{3})$?

Recall

tan plane requires

① pt plane ✓

② vector ⊥ to plane

$u=2$

$2v^2=2 \rightarrow v^2=1 \rightarrow v=\pm 1$

$u^2+v=3$

$\cancel{2^2+1=3}$

or

$2^2-1=3$

$\Rightarrow \boxed{u=2,\ v=-1}$

Vector ⊥ to plane: $\vec{n} = \underset{\frac{\partial \vec{r}}{\partial u}}{\vec{r}_u} \times \underset{\frac{\partial \vec{r}}{\partial v}}{\vec{r}_v}$

$= \langle 1, 0, 2u \rangle \times \langle 0, 4v, 1 \rangle$

$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{bmatrix}$

$= \langle -8uv, -(1-0), 4v \rangle$

$= \langle -8uv, -1, 4v \rangle$

$u=1, v=1$

$\vec{r}(1,1) = \langle 1,2,2 \rangle$

$\vec{r}_u = \langle 1,0,2 \rangle$

$\vec{r}_v = \langle 0,4,1 \rangle$

$r_u \times r_v = \langle -8,-1,4 \rangle$

at $u=2, v=-1$

$\vec{r}_u = \langle 1,0,4 \rangle$

$\vec{r}_v = \langle 0,-4,1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 16,-1,-4 \rangle$

↓ Tan plane

$16(x-2) - (y-2) - 4(z-3) = 0$

Surface area:

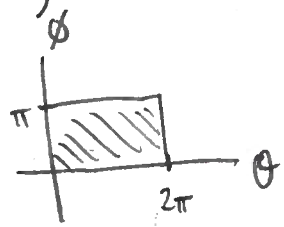
parametric surface $\begin{cases} \vec{r}(u,v) \\ u,v \in D \end{cases}$

$$\delta A = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA$$

Ex: Find surface area of a sphere of radius a.

Soln: Parametrize using spherical coordinates

$$\begin{cases} \vec{r}(\theta,\phi) = \langle a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos\phi \rangle \\ \left.\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{matrix}\right\} D: \end{cases}$$



So,

$$\boxed{\text{Surface Area} = \iint_D \|\vec{r}_\theta \times \vec{r}_\phi\| \, dA}$$

there must be an error here somewhere

$$\vec{r}_\theta \times \vec{r}_\phi = \langle a\,s(\phi)(-s(\theta)), a\,s(\phi)c(\theta), 0 \rangle \times \langle a\,c(\phi)c(\theta), a\,c(\phi)s(\theta), -a\,c(\phi) \rangle$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\,s(\phi)s(\theta) & a\,s(\phi)c(\theta) & 0 \\ a\,c(\phi)c(\theta) & a\,c(\phi)s(\theta) & -a\,c(\phi) \end{bmatrix}$$

$$-a^2 s(\phi)c(\phi)\underbrace{\cancel{[s^2(\theta)+c^2(\theta)]}}_{=1}$$

$$= \langle -a^2 s(\phi)c(\theta)c(\phi), -(a^2 s(\phi)s(\theta)c(\phi) - 0), -a^2 s(\phi)c(\phi)s^2(\theta) - a^2 s(\phi)c(\phi)c^2(\theta) \rangle$$

$$\|\vec{r}_\theta \times \vec{n}_\phi\| = \sqrt{a^4 s^2(\phi) c^2(\theta) c^2(\phi) + a^4 s^2(\phi) s^2(\theta) c^2(\phi) + a^4 s^2(\phi) c^2(\phi)}$$

$$= \sqrt{a^4 s^2(\phi) c^2(\phi) \underbrace{\left(c^2(\theta) + s^2(\theta)\right)}_{=1} + a^4 s^2(\phi) c^2(\phi)}$$