Theorem: If $\vec{F}$ is a vector field on a simply connected domain whose component functions have ctn partial derivatives and $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

Ex: Show that

$$\vec{F} = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$$

is conservative + find its potential.

Note: $\vec{F}$ is defined on all of $\mathbb{R}^3$ — which is simply cnctd.

Soln: Compute

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{bmatrix}$$

$$= \langle 6xyz^2 - 6xyz^2, -(3y^2z^2 - 3y^2z^2), 2yz^3 - 2yz^3 \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

Find potential for $\vec{F}$, i.e. find f s.t. $\vec{F} = \nabla f$ ②

$= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$$\begin{cases} \frac{\partial f}{\partial x} = y^2 z^3 & \text{(i)} \checkmark \\ \frac{\partial f}{\partial y} = 2xyz^3 & \text{(ii)} \checkmark \\ \frac{\partial f}{\partial z} = 3xy^2z^2 & \text{(iii)} \end{cases}$$

$$\text{(i)} \to \boxed{f = xy^2z^3 + g(y,z)}$$

$$2xyz^3 \underset{\text{(ii)}}{=} \frac{\partial f}{\partial y} \underset{\text{compute}}{=} 2xyz^3 + \frac{\partial g}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 0 \Rightarrow \boxed{g = h(z)}$$

We now have

$$\boxed{f = xy^2z^3 + h(z)}$$

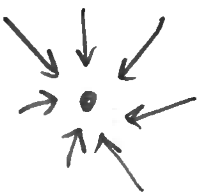
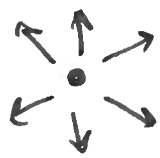
$$\cancel{3xy^2z^2} \underset{\text{(iii)}}{=} \frac{\partial f}{\partial z} \underset{\text{Computed}}{=} \cancel{3xy^2z^2} + h'(z)$$

$$h'(z) = 0 \to h = C, \text{ where } C \text{ is constant}$$

Therefore ~~the~~ a potential of $\vec{F}$ is

$$\boxed{f(x,y,z) = xy^2z^3 + C}$$

Divergence of a vector field

(div < 0)	(div > 0)	div = 0
Sinks	Sources	
		 ("incompressible")

Def: The divergence takes a vector field on $\mathbb{R}^3$ to a scaler function:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$
$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$
$$= P_x + Q_y + R_z$$

Ex: Let $\vec{F} = \langle xz, xyz, -y^2 \rangle$

compute $\operatorname{div} \vec{F}$.

Soln: $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xz, xyz, -y^2 \rangle$$
$$= z + xz + 0$$
$$= z + xz$$

$(\operatorname{div} \vec{F})(0,0,2) = 2 \sim$ source

$(\operatorname{div} \vec{F})(0,0,-2) = -2 \sim$ sink

(see visually w/ calcplot3D)

Theorem: $F=\langle P,Q,R\rangle$ (assume P,Q,R have ctn partials)

$$\operatorname{div}(\operatorname{curl}(\vec{F}))=0$$

Proof:

$$\nabla\cdot(\nabla\times\vec{F})=\left\langle \frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\cdot\overbrace{\langle R_y-Q_z,\,P_z-R_x,\,Q_x-P_y\rangle}^{\text{curl }\vec{F}}$$

$$=(\cancel{R_{yx}}-\cancel{Q_{zx}})+(\cancel{P_{zy}}-\cancel{R_{xy}})+(\cancel{Q_{xz}}-\cancel{P_{yz}})$$

$$=0$$

Ex: Show that $\vec{F}=\langle xz,\,xyz,\,-y^2\rangle$ can NOT be written as curl of another vector field, i.e.

$$\vec{F}\neq\operatorname{curl}\vec{G} \text{ for any } \vec{G}$$

Soln: Compute

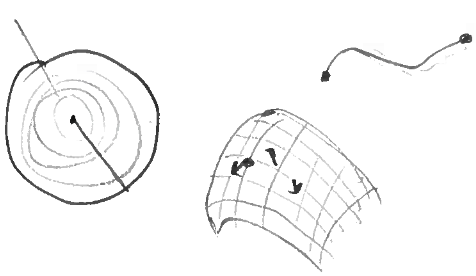
$$\operatorname{div}\vec{F}=z+xz\neq 0$$

So by theorem, we conclude desired result.

Note: often $\operatorname{div}(\nabla f)$ is computed, so we give it special notation

$$\nabla^2 f=\operatorname{div}(\nabla f)=(\nabla\cdot\nabla)f$$

$$=\underbrace{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}}_{\text{"the Laplacian"}}$$



defines "harmonic functions"