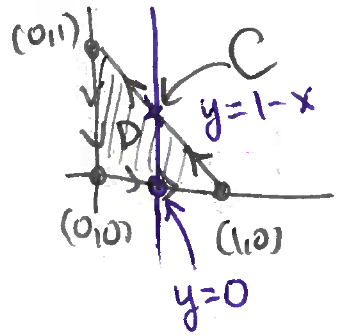
Green's Thm $\vec{F} = \langle P, Q \rangle$, C closed curve, D is region bdd by C

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

Ex (continued)

$\vec{F} = \langle x^4, xy \rangle$

$P = x^4$, $Q = xy$



last time

$$\int_C \vec{F} \cdot d\vec{r} = \frac{1}{6}$$

now:

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iint_D y - 0 \, dA$$

$$= \int_0^1 \int_0^{1-x} y \, dy \, dx$$

$$= \int_0^1 \left. \frac{y^2}{2} \right|_{y=0}^{y=1-x} dx$$

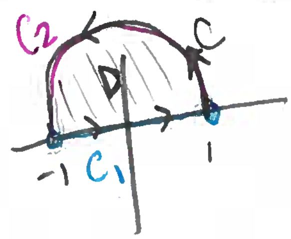
$$= \int_0^1 \frac{(1-x)^2}{2} dx = \frac{1}{2} \int_0^1 1 - 2x + x^2 \, dx$$

$$= \frac{1}{2} \left[\cancel{1} - \cancel{1} + \frac{1}{3} \right]$$

$$= \frac{1}{6}$$

Ex: $\displaystyle\int_C \underbrace{y^2\,dx + 3xy\,dy}$ where C is upper half unit circle

$\vec{F} = \langle y^2, 3xy \rangle$



Soln: Compute line integral directly:

parametrize C_1

line seg from $(-1,0)$ to $(1,0)$

$$\begin{cases} \vec{r}_1(t) = t\langle 1,0\rangle + (1-t)\langle -1,0\rangle \\ \quad = \langle t-1+t, 0\rangle \\ \quad = \langle 2t-1, 0\rangle \\ 0 \le t \le 1 \end{cases}$$

parametrize C_2

$$\begin{cases} \vec{r}(t) = \langle \cos(t), \sin(t)\rangle \\ 0 \le t \le \pi \end{cases}$$

Compute

$$\int_C y^2\,dx + 3xy\,dy = \int_{C_1} y^2\,dx + 3xy\,dy + \int_{C_2} y^2\,dx + 3xy\,dy$$

$\cos^2 + \sin^2 = 1$

Correction in green

$$= \int_0^1 0^2(2) + 3(2t-1)(0)(0)\,dt$$

$u = \cos t$

$du = -\sin(t)\,dt$

$$+ \int_0^\pi \sin^2(t)(-\sin(t)) + 3\cos^2(t)\sin(t)\,dt$$

$\sin^3(t)$

$= \sin^2(t)\sin(t)$

$= (1-\cos^2(t))\sin(t)$

$= \sin(t) - \cos^2(t)\sin(t)$

$u = \sin t$

$du = \cos t\,dt$

$$= 0 - \int_0^\pi \sin^3(t)\,dt + 3\int_0^\pi \cos^2(t)\sin(t)\,dt$$

$$= -\int_0^\pi \sin(t)\,dt + \int_0^\pi \cos^2(t)\sin(t)\,dt + 3\int_0^\pi \cos^2(t)\sin(t)\,dt$$

$-\cos(t)$

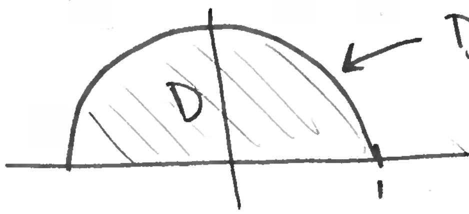
$$= -\left[-\cos(\pi) - (-\cos(0))\right] + 4\int_1^{-1} u^2\,du + \cancel{3\int_0^0 u^2\,du}$$

$$= -[1+1] + 4\left[\frac{u^3}{3}\Big|_{-1}^{1}\right]$$

$$= -2 + 4\left(\frac{1}{3} - \frac{(-1)^3}{3}\right)$$

$$= -2 + 4\left(\frac{2}{3}\right) = \cancel{-\frac{4}{3}}\ \frac{2}{3}$$

Do same w/ Green's theorem:



$$D = \left\{(r,\theta) : \begin{matrix} 0 < r < 1 \\ 0 \le \theta \le \pi \end{matrix}\right\}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\int_C \underbrace{y^2}_{P}\,dx + \underbrace{3xy}_{Q}\,dy \underset{\text{Green's}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \iint_D \underbrace{3y - 2y}_{=y}\, dA$$

$$= \int_0^{\pi}\int_0^1 r\sin(\theta)\ \underset{\text{extra}}{r}\ dr\,d\theta$$

$$= \int_0^{\pi} \sin(\theta)\left(\frac{1}{3}\right) d\theta$$

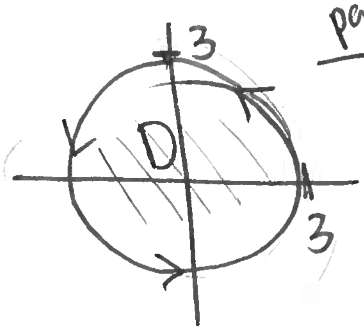
$$= \frac{1}{3}\left[-\cos(\theta)\Big|_0^{\pi}\right] = \frac{1}{3}\left[-\cos(\pi) - (-\cos(0))\right]$$

$$= \frac{1}{3}\left[1+1\right] = \frac{2}{3}$$

Ex: Evaluate $\int_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$

where C is circle $x^2+y^2=9$.

Soln: Try directly w/ magic formula:



parametrize C

$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$

$0 \le t \le 2\pi$

x y

$$\int_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$$

$$= \int_0^{2\pi} \left(9\sin(t) - e^{\sin(3\cos t)}\right)\left(-3\sin(t)\right) + \left(21\cos(t) + \sqrt{81\sin^4(t)+1}\right)3\cos(t)\,dt$$

NOPE! We can't integrate this mess.

By Green's thm

$$\int_C \underbrace{\left(3y - e^{\sin x}\right)}_{P}dx + \underbrace{\left(7x + \sqrt{y^4+1}\right)}_{Q}dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_D 7 - 3\,dA$$

$$= 4\underbrace{\iint_D 1\,dA}_{\text{Area}(D)} = 4 \cdot 9\pi = 36\pi$$

We can compute areas with Green's thm

Since $\text{Area}(D) = \iint_D 1 \, dA$, just need to pick $\vec{F} = \langle P, Q \rangle$

such that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1.$

LOTS of options

$$\begin{cases} P = 0 \\ Q = x \end{cases} \qquad \begin{cases} P = -y \\ Q = 0 \end{cases} \qquad \begin{cases} P = -\frac{1}{2}y \\ Q = \frac{1}{2}x \end{cases}$$

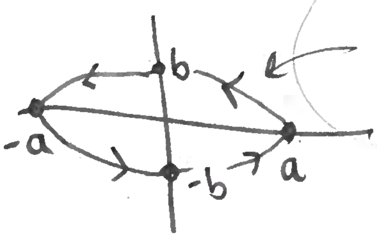
$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = \dfrac{1}{2} + \dfrac{1}{2} = 1$

$$\text{Area} = \oint_C x \, dy \qquad \text{Area} = -\oint y \, dx \qquad \text{Area} = \frac{1}{2}\oint_C \underbrace{x}_{Q} \, dy - \underbrace{y}_{P} \, dx$$

$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

Ex: Find area of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Soln:



Parametrize

$\vec{r}(t) = \langle$