Ex: If $\vec{F} = \langle y^2, 2xy + e^{3z} [redacted], 3ye^{3z} \rangle$

find its potential function f.

$\vec{F} = \nabla f$

Soln: Find $f \sim \nabla f = \vec{F}$

$\frac{\partial f}{\partial x} = y^2$ (i)

$\frac{\partial f}{\partial y} = 2xy + e^{3z}$ [redacted] (ii)

$\frac{\partial f}{\partial z} = 3ye^{3z}$ (iii)

From (i)

$$f = \int y^2\,dx = xy^2 + g(y,z)$$

By (ii), compute

$$\cancel{2xy} + e^{3z} = \frac{\partial f}{\partial y} = \cancel{2xy} + \frac{\partial g}{\partial y}(y,z)$$

given computed

$$\frac{\partial g}{\partial y} = e^{3z}$$

$$g = \int \frac{\partial g}{\partial y}\,dy = \int e^{3z}\,dy = ye^{3z} + h(z)$$

$$\Rightarrow \boxed{f = xy^2 + ye^{3z} + h(z)}$$

$$3ye^{3z} = \frac{\partial f}{\partial z} = 3ye^{3z} + h'(z) \Longrightarrow h'(z) = 0$$

computed

given (iii)

$$\rightarrow h(z) = \int 0\,dz = C$$

$$\boxed{f = xy^2 + ye^{3z} + C}$$

Green's Theorem

There is a relationship b/w $\iint_D \cdot \, dA$ and $\int_C \cdot \, dx + \cdot \, dy$

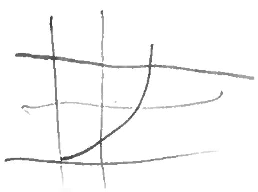
(video: planimeters)

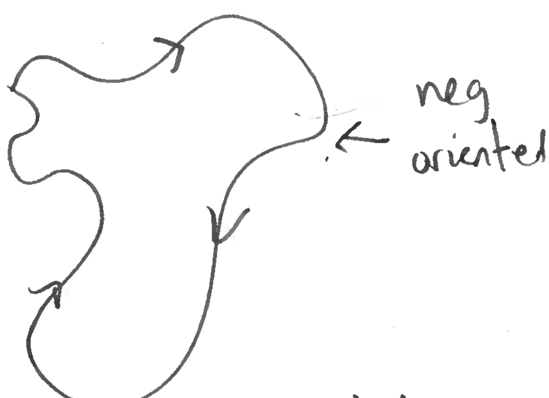
no self ∩

start pt & end pt same

Def: If C is a simple closed curve, we say C has positive orientation if the curve "goes counterclockwise"

Similarly, negative orientation if it goes clockwise.





Green's Theorem: Let C be a pos. oriented, piecewise smooth, simple closed curve and let D be a simple region bounded by C. If P, Q have ctn partial derivatives in D, then if $\vec{F} = \langle P, Q \rangle$

line int

dbl int

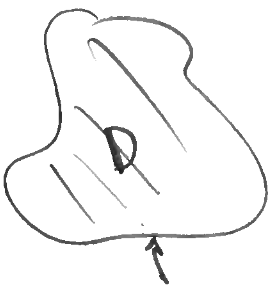
$$\int_C P\,dx + Q\,dy = \int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

Some notation

$\oint_C$ ~ means line int w/ C ⊕ oriented

Sometimes D ← region, C ← curve

"∂D" ← in topology this notation is used for "boundary of D"



$C = \partial D$

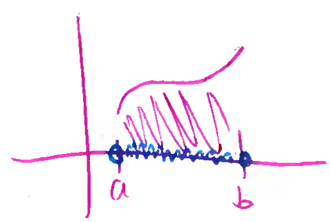
COMPARE

Green's Thm $\quad \int_{\partial D} \underbrace{P\,dx + Q\,dy}_{\text{not-differentiated}} = \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{differentiation}}\,dA$

"inside"

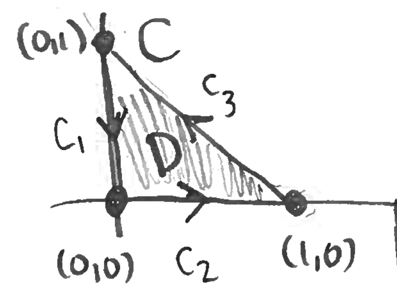
Fund Thm Calc $\quad \int_{\{a,b\}} f = \overbrace{f(b) - f(a)}^{} = \int_a^b \widetilde{f'(x)}\,dx$

boundary



Ex: Evaluate $\int_C x^4\,dx + xy\,dy$ where C is triangle w/ vertices $(0,0)$, $(1,0)$, and $(0,1)$

Soln:



First compute $\int_C x^4\,dx + xy\,dy$ directly.

Parametrize C_1, C_2, C_3

C_1

$\vec{r}_1(t) = t\langle 0,0\rangle + (1-t)\langle 0,1\rangle$
$= \langle 0, 1-t\rangle$
$0 \le t \le 1$

C_3

$\vec{r}_3(t) = t\langle 0,1\rangle + (1-t)\langle 1,0\rangle$
$= \langle 1-t, t\rangle$
$0 \le t \le 1$

$\int_C f\,dx = f(\vec{r}(t)) \cdot x'$

C_2

$\vec{r}_2(t) = t\langle 1,0\rangle + (1-t)\langle 0,0\rangle$
$= \langle t, 0\rangle$
$0 \le t \le 1$

So,

$$\int_C x^4\,dx + xy\,dy = \int_{C_1} x^4\,dx + xy\,dy + \int_{C_2} x^4\,dx + xy\,dy + \int_{C_3} x^4\,dx + xy\,dy$$

$$= \overbrace{\int_{C_1} 0^4\,dx + 0\,dy}^{0} + \int_{C_2} t^4 \cdot dx + 0\,dy + \int_{C_3} (1-t)^4\,dx + (t-t^2)\,dy$$

$$= \int_0^1 t^4\,dt + \int_0^1 (1-t)^4(-1) + (t-t^2)1\,dt \quad u = 1-t \quad du = -dt$$

$$= \frac{t^5}{5} + \frac{u^5}{5}\Big|_1^0 + \frac{t^2}{2} - \frac{t^3}{3}\Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{5} + \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}$$

Next time ⇝ do this w/ Green's thm