Recall: $\vec{F}$ conservative means $\vec{F} = \nabla f$

If $\vec{F}$ conservative, $\vec{F} = \langle P, Q \rangle$, then there is a scalar function f such that

$$\nabla f = \boxed{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle P, Q \rangle} = \vec{F}$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = P \\[2ex] \dfrac{\partial f}{\partial y} = Q \end{cases}$$

(Clairaut's Theorem)
Recall: if f is sufficiently smooth ~ $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

$$\begin{cases} \dfrac{\partial f}{\partial x} = P \\ \dfrac{\partial f}{\partial y} = Q \end{cases} \longrightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \overset{\text{Clairaut}}{=} \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial Q}{\partial x}$$

contrapositive

if P, then Q

$\boxed{\text{SAME}}$

if not Q, then not P

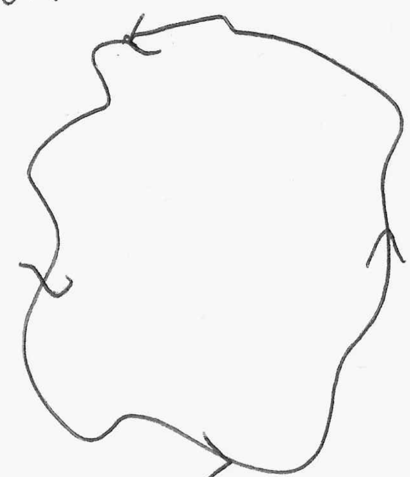
$$\Rightarrow \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \longleftarrow \left(\text{similar to } ~~\text{Riemann}~~ \text{ Cauchy-Riemann}\right)$$

Theorem: If $\vec{F} = \langle P, Q \rangle$ is conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
(assumption) (conclusion)

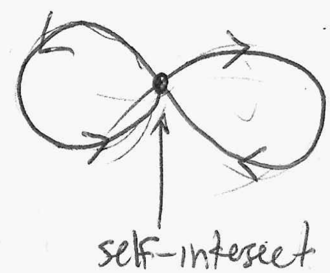
Question: Is it true that if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ is conservative?

Answer: NO — needs an explanation

Def: A curve C is called a **simple curve** if it does not self-intersect.

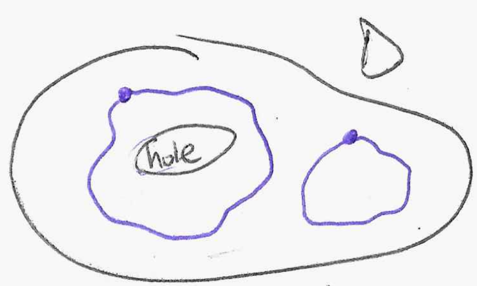


simple

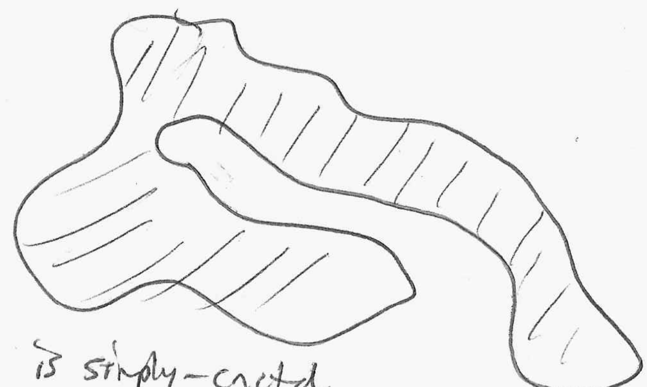


not simple.

Def: A **simply-connected region** D means that ~~all~~ D has no holes.



not simply cnctd
(has hole)



is simply-cnctd

$\frac{1}{x}$

<u>Theorem</u>: If $\vec{F} = \langle P, Q \rangle$ is a vector field on a open, simply-connected region D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ is conservative.

<u>Ex</u>: Conservative or not?

defined on whole plane $\mathbb{R}^2$ ~ <u>simply cnctd</u>

$$\vec{F} = \langle \underbrace{x-y}_{P}, \underbrace{x-2}_{Q} \rangle$$

Calculate

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1 \implies \vec{F} \text{ not conservative}$$

<u>aren't equal!</u>

Ex: $\vec{F} = \langle \underbrace{2xe^{xy} + x^2ye^{xy}}_{P}, \overbrace{x^3e^{xy} + 2y}^{Q} \rangle$

defined on $\mathbb{R}^2$ ~ simply connected

$$\frac{\partial P}{\partial y} = 2x^2e^{xy} + x^2\left[e^{xy} + ye^{xy}x\right] = 2x^2e^{xy} + x^2e^{xy} + yx^3e^{xy}$$

$$\frac{\partial Q}{\partial x} = 3x^2e^{xy} + x^3e^{xy}y$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \boxed{\vec{F} \text{ is conservative}}$$

Ex: Conservative or not? If so, find potential f.

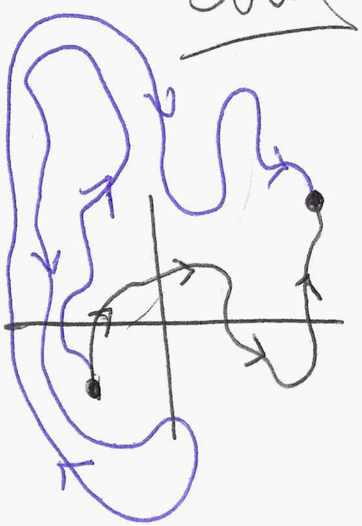
$\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F} = \langle \underbrace{3+2xy}_{P}, \underbrace{x^2 - 3y^2}_{Q} \rangle$$

$\exists f \quad \vec{F} = \nabla f$

$\langle P, Q \rangle = \langle f_x, f_y \rangle$

Soln: $\dfrac{\partial P}{\partial y} = 2x = \dfrac{\partial Q}{\partial x} \Longrightarrow \vec{F}$ conservative $\longleftrightarrow \exists f \quad \vec{F} = \nabla f$



$$\begin{cases} \dfrac{\partial f}{\partial x} = P = 3+2xy & \text{(i)} \\ \dfrac{\partial f}{\partial y} = Q = x^2 - 3y^2 & \text{(ii)} \end{cases}$$

(i) $\longrightarrow$ $\boxed{f = \int 3+2xy \, dx + C(y)}$

$\boxed{= 3x + x^2 y + C(y)}$

$$\boxed{x^2 - 3y^2 \overset{\text{(ii)}}{=} \frac{\partial f}{\partial y} = x^2 + C'(y)}$$

$$\cancel{x^2} - 3y^2 = \cancel{x^2} + C'(y)$$

no x on it

$$C'(y) = -3y^2 \longrightarrow C(y) = \int -3y^2 \, dy = -y^3 + D$$

Therefore $\boxed{f = 3x + x^2 y - y^3 + D}$ $\longrightarrow$ Check

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 3+2xy, x^2 - 3y^2 \rangle = \vec{F} \quad ✓$$