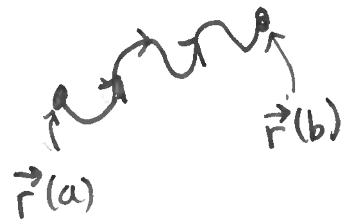
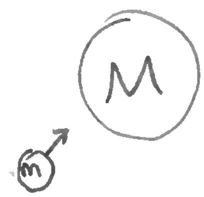
**FTOLI**: $C$ parametrized by $\begin{cases}\vec{r}(t)\\ a \le t \le b\end{cases}$

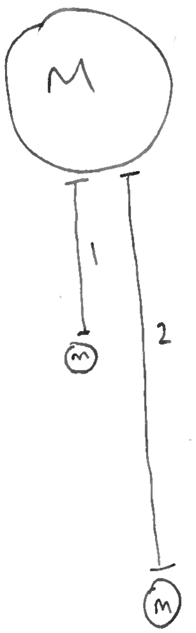



$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

**Ex**: (Work) Find work done by gravitational field $\vec{F}(\vec{x}) = -\dfrac{mMG}{\|\vec{x}\|^3}\vec{x}$ ← $\langle x, y, z\rangle$




when moving a particle w/ mass $m$ from $(3,4,12)$ to $(2,2,0)$.




**Sol'n**: Our goal: find $f$ such that

$$\nabla f = \vec{F}$$

Turns out using

$$f = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$ will work.

"potential" → $f$

**BECAUSE**: Compute

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = mMG\left\langle \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}, \frac{\partial}{\partial y}(\;)^{-1/2}, \frac{\partial}{\partial z}(\;)^{-1/2} \right\rangle$$

$$= mMG\left\langle -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x), -\tfrac{1}{2}(\;)^{-3/2}(2y), -\tfrac{1}{2}(\;)^{-3/2}(2z) \right\rangle$$

$$= \frac{-mMG}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\langle x, y, z\rangle$$

$\vec{x} = \langle x, y, z\rangle$

$$= \frac{-mMG}{\|\vec{x}\|^3}\vec{x}$$

Therefore,

$$\text{Work} = \int_{C} \vec{F} \cdot d\vec{r} = \int_{C} \nabla f \cdot d\vec{r}$$

Curve from $(3,4,12)$ to $(2,2,0)$

$$\overset{\text{FTOLI}}{=} f(2,2,0) - f(3,4,12)$$

$$= \frac{-mMG}{\sqrt{4+4+0}} - \frac{(-mMG)}{\sqrt{9+16+144}}$$

$$= -mMG\left(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{169}}\right)$$

---

Ex: Compute $\int_C \nabla f \cdot d\vec{r}$ where

$$f = \cos(\pi x) + \sin(\pi y) - xyz$$

and $C$ is segment from $(1, \frac{1}{2}, 2)$ to $(2,1,-1)$.

Soln: Using FTOLI,

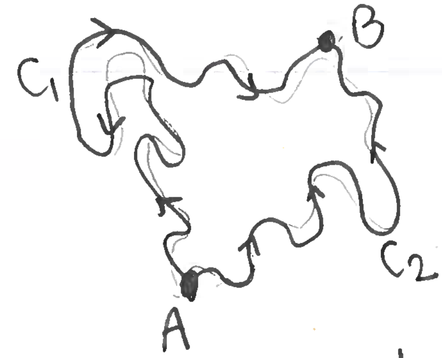
$$\int_C \nabla f \cdot d\vec{r} = f(2,1,-1) - f(1,\tfrac{1}{2},2)$$

$$= \left[\cos(2\pi) + \sin(\pi) - (-2)\right] - \left[\cos(\pi) + \sin\left(\tfrac{\pi}{2}\right) - 1\right]$$

$$= (1 + 0 + 2) - (\cancel{-1+1} - 1)$$

$$= 4$$

# Independence of path




$C_1$ and $C_2$ are two different paths from A to B

Recall: earlier we saw an example showing that, in general,

$$\int_C \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r} \qquad \vec{F} = \langle y^2, x \rangle$$

(23-26 October notes) $\rightarrow$ $\left( ex: \int_{C_1} y^2\,dx + x\,dy \neq \int_{C_2} y^2\,dx + x\,dy \right)$

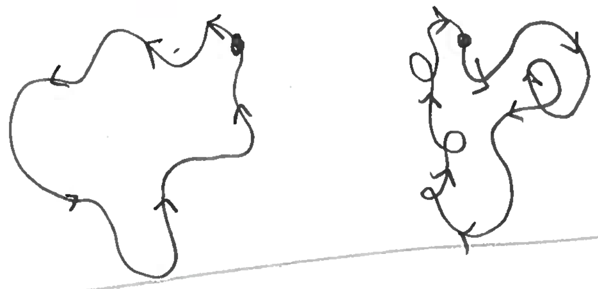
$$-\frac{5}{6} \neq \frac{209}{6}$$

FTOLI: $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$

So conclude: we cannot write $\vec{F} = \langle y^2, x \rangle$ as $\vec{F} = \nabla f$.

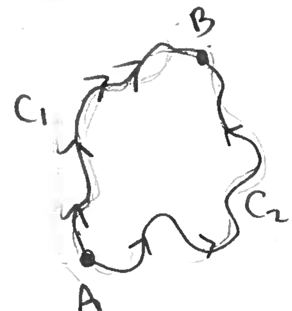
Def: We say $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for all paths $C_1, C_2$ going from A to B

**Def**: A curve C is called <u>closed</u> if "end point" = "start point"



**Theorem**: $\int_C \vec{F}\cdot d\vec{r}$ is indep of path iff $\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed paths $C$

Why?




$\int_{C_1} = \int_{C_2}$

$\Downarrow$

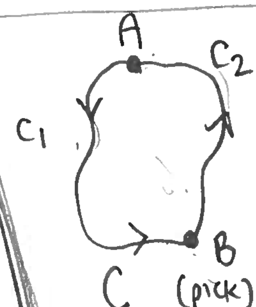
$\int_{C_1} - \int_{C_2} = 0$

$\Downarrow$

$C_1 \cup (-C_2)$ is a closed path

$\Downarrow$

$$\int_{C_1 \cup (-C_2)} = \int_{C_1} + \int_{-C_2} = \int_{C_1} - \int_{C_2} = 0$$




$\int_C = 0$

$C_1 \sim A \to B$

$C_2 \sim B \to A$

$-C_2 \sim A \to B$

$0 = \int_C = \int_{C_1} + \int_{C_2}$

$= \int_{C_1} - \int_{-C_2}$

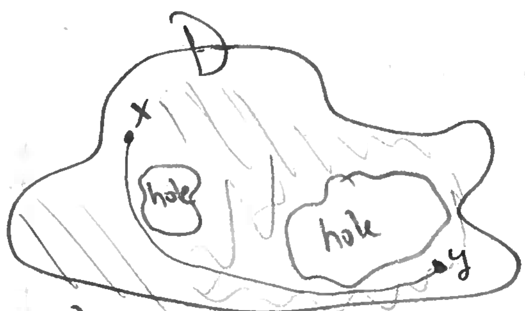
$\Rightarrow \int_{C_1} = \int_{-C_2}$

✓

# Topological definitions


**Def**: D is an open set means for any $x \in D$, there is a disk around x lying entirely within D

**Def**: D is (path) connected means for any $x, y \in D$, there is a path from x to y lying entirely in D




(path) connected


no path from x to y lying entirely in D

$\Rightarrow$ not (path) connected

note: "connected" and "path connected" aren't same in general:

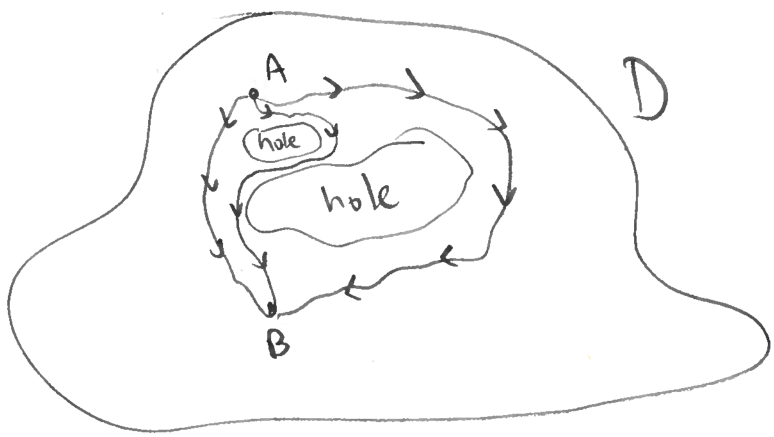
$y = \sin\left(\frac{1}{x}\right)$

$$\underbrace{\left\{(x,y) : \begin{matrix} y = \sin\left(\frac{1}{x}\right) \\ 0 < x < 1 \end{matrix}\right\} \cup \overbrace{\left\{(x,y) : \begin{matrix} x = 0 \\ -1 \le y \le 1 \end{matrix}\right\}}^{\text{"limit bar"}}}_{\text{"topologist's sine curve"}} \sim \text{connected but not path connected}$$

impossible to find a finite length path from a pt on limit bar to another pt off the limit bar

**Theorem:** Suppose $\vec{F}$ is a continuous vector field on an open (path) connected region $D$.

If $\int_C \vec{F} \cdot d\vec{r}$ is indep. of path, then $\vec{F}$ is conservative (i.e. $\vec{F} = \nabla f$ for some $f$).




**Question:** How can we check if a given $\vec{F}$ is conservative?