We've seen

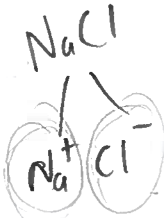
scaler line int ~ $\int_C f\, ds$

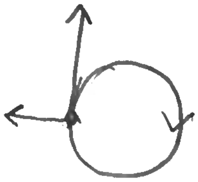
vector field line int ~ $\int_C \vec{F}\cdot d\vec{r}$ $= \int_C F(\vec{r}(t))\cdot \vec{r}'(t)$

("circulation") → C

when C is a "closed curve"

Flux – how much material does the vector field push thru C? — material crossing a boundary





flux line integral: unit vector

$$\int_C \vec{F}(\vec{r}(t))\cdot \boxed{\frac{\vec{n}}{\|\vec{n}\|}}\, dt \text{ where } \vec{n} = \langle y', -x'\rangle$$

Ex: $\vec{F} = \langle -x, -y \rangle$ $\vec{r} = \langle \overset{x}{\cos(t)}, \overset{y}{\sin(t)} \rangle$

$0 \leq t \leq 2\pi$

$\vec{n} = \langle y', -x' \rangle$

$= \langle \cos(t), \sin(t) \rangle$

$= \langle x, y \rangle$

$= \langle \cos t, \sin t \rangle$

$\|\vec{n}\| = 1$

flux →

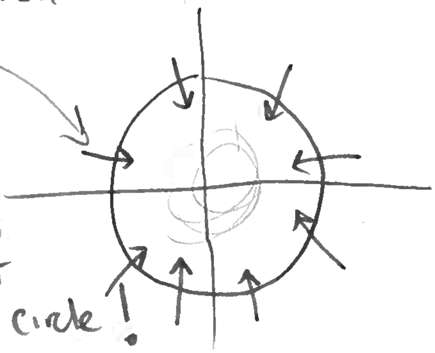
FLUX

$$\int_C \vec{F} \cdot \frac{\vec{n}}{\|\vec{n}\|}\, dt$$

$$= \int_0^{2\pi} \langle -\cos(t), -\sin(t) \rangle \cdot \langle \cos(t), \sin t \rangle\, dt$$

$$= -\int_0^{2\pi} \cos^2(t) + \sin^2(t)\, dt$$

$$= \boxed{-2\pi}$$



meens the field is moving 2π worth of material into the circle!

(negative → inwards)

$\vec{r}\,' = \langle -\sin(t), \cos(t) \rangle$

$= \langle -y, x \rangle$

↓ (circulation)

$$\int_C \vec{F} \cdot \vec{r}\,'\, dt = \int_0^{2\pi} \langle -\cos(t), -\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle\, dt$$

$$= \int_0^{2\pi} + \cos(t)\sin(t) - \sin(t)\cos(t)\, dt$$

$$= \int 0 = 0$$

↑ circulation

Ex: $\vec{F} = \langle x, y \rangle$

$\begin{cases} \vec{r}(t) = \langle \overset{x\downarrow}{\cos t}, \overset{y\downarrow}{\sin t} \rangle \\ 0 \le t \le 2\pi \end{cases}$

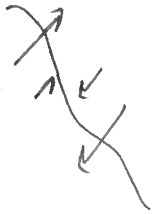
$\vec{n} = \langle y', -x \rangle$

$= \langle \cos t, \sin t \rangle$

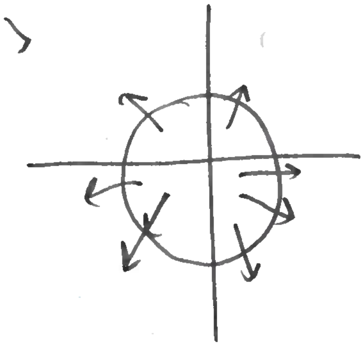
$\|\vec{n}\| = 1$

flux

$$\text{flux} = \int_C \vec{F} \cdot \left(\frac{\vec{n}}{\|\vec{n}\|} \right)$$



$$= \int_C \langle \cos t, \sin t \rangle \cdot \langle \cos t, \sin t \rangle$$



$$= \int_0^{2\pi} 1 \, dt = 2\pi$$

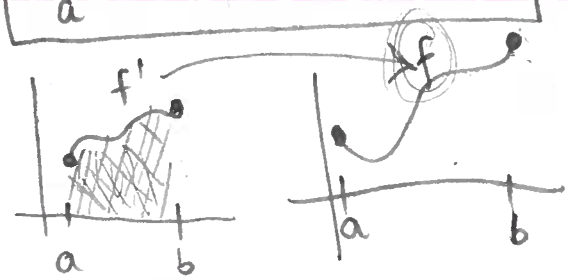
(positive flux → outwards)

Fundamental Theorem of Line Integrals

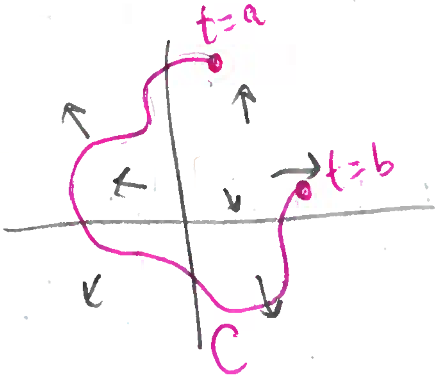
Calc 1 : fund. thm. of calculus

$$\frac{d}{dx}\int^{x} f(t)\,dt = f(x)$$

$$\boxed{\int_a^b f'(t)\,dt = \textcircled{f}(b) - f(a)}$$



Calc 3 :

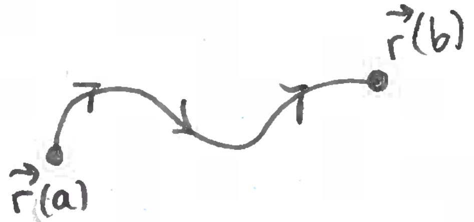


We can calculate

$$\int_C \vec{F}\cdot d\vec{r} \text{ in a similer way}$$

as FTC from Calc 1 ~ using only endpoints of curve C

Curve C parametrized by $\begin{cases} \vec{r}(t) = \langle x(t), y(t) \rangle \\ a \le t \le b \end{cases}$

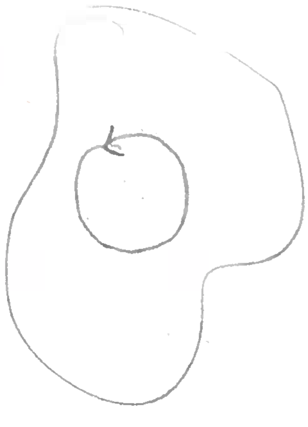


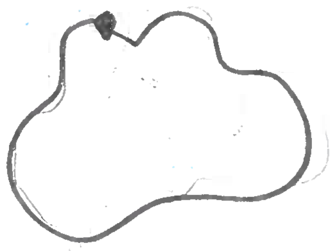
FTOLI: $$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

gradient field of a "potential funct" called f

$f = x^2 y$
$\nabla f = \langle 2xy, x^2 \rangle$

What happens if C is a closed curve?
i.e. if $\vec{r}(a) = \vec{r}(b)$?





$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(a)) - f(\vec{r}(b)) = 0$$