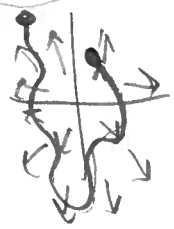
✓ scalar line $\int$: $\int f\,ds$ — use — mass of wire

vector field line $\int$: $\int \vec{F}\cdot d\vec{r}$ — work

$W = F \cdot d$



Ex: Find work done in moving a particle along quarter circle $\vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle$, $0 \le t \le \frac{\pi}{2}$ in field $\vec{F} = \langle x^2, -xy \rangle$

$\vec{r}\,' = \langle -\sin(t), \cos(t) \rangle$

Soln: Compute

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \langle \cos^2(t), -\cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt$$

$$= \int_0^{\pi/2} -\sin(t)\cos^2(t) - \cos^2(t)\sin(t)\, dt$$

$-\int_a^b = \int_b^a$

$u = \cos t$

$du = -\sin(t)\,dt$

$$= -2\int_0^{\pi/2} \sin(t)\cos^2(t)\, dt$$

$$= 2\int_1^0 u^2\, du = -2\int_0^1 u^2\, du = -\frac{2}{3}$$

Ex: Find work done by $\vec{F} = \langle yx^2, y^2 \rangle$ when moving particle from $(0,0)$ to $(1,1)$ along parabola $y = x^2$.

Soln:

↓ parametrize

$x \downarrow \quad y \downarrow$

$$\begin{cases} \vec{r}(t) = \langle t, t^2 \rangle \\ 0 \le t \le 1 \end{cases}$$

So, $\vec{r}\,'(t) = \langle 1, 2t \rangle$

Compute

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle t^4, t^4 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_0^1 t^4 + 2t^5 \, dt$$

$$= \frac{1}{5} + \frac{2}{6}$$

Ex: Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=\langle xy, yz, zx\rangle$ and C is twisted cubic (x, y, z)

$$\begin{cases}\vec{r}(t)=\langle t, t^2, t^3\rangle \\ 0\le t\le 1\end{cases}$$

Soln: $\vec{r}\,'=\langle 1, 2t, 3t^2\rangle$

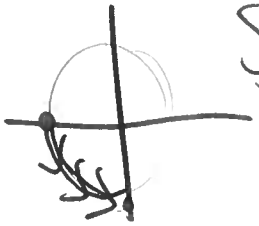
Compute

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle t^3, t^5, t^4\rangle\cdot\langle 1, 2t, 3t^2\rangle\, dt$$

$$=\int_0^1 t^3+2t^6+3t^6\, dt$$

$$=\left[\frac{t}{4}+\frac{2}{7}+\frac{3}{7}\right]=\frac{1}{4}+\frac{5}{7}$$

Ex: Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=\langle 0, x\rangle$ and C is qtr circle of radius 1 in QIII.



Soln:

$$\begin{cases}\vec{r}=\langle \cos t, \sin t\rangle \\ \pi\le t\le \frac{3\pi}{2}\end{cases}$$

$$\vec{r}\,'=\langle -\sin t, \cos t\rangle$$

Compute

$$\int_C \vec{F}\cdot d\vec{r} = \int_\pi^{3\pi/2}\langle 0, \cos t\rangle\cdot\langle -\sin t, \cos t\rangle\, dt$$

$$=\int_\pi^{3\pi/2}\cos^2(t)\, dt = \int_\pi^{3\pi/2}\frac{1+\cos(2t)}{2}\, dt$$

$$=\frac{1}{2}\left(\frac{3\pi}{2}-\pi\right)+\frac{1}{2}\cdot\frac{1}{2}\sin(2t)\Big|_\pi^{3\pi/2}=\frac{\pi}{4}+\frac{1}{4}(0-0)=\frac{\pi}{4}$$