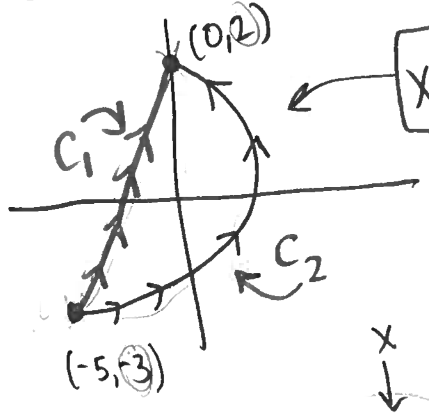
Continued from 23 October

$y = f(\overset{\downarrow}{x})$

$\vec r(t) = \langle t, f(t)\rangle$

<u>on C_2</u> parametrize C_2



$\boxed{x = 4 - y^2}$

$$\Rightarrow \begin{cases} \vec r(t) = \langle \overset{x}{4-t^2}, \overset{y}{t} \rangle \\ -3 \le t \le 2 \end{cases}$$

$$\Rightarrow \vec r'(t) = \langle \underset{\text{"}x'(t)\text{"}}{-2t}, \underset{\text{"}y'(t)\text{"}}{1} \rangle$$

So,

$$\int_C y^2 dx + x\,dy = \int_{-3}^{2} t^2 \overset{-2t^3}{(-2t)} + (4-t^2)(1)\;dt$$

$$= \int_{-3}^{2} -2t^3 - t^2 + 4\;dt = -\frac{2}{4}t^4 - \frac{1}{3}t^3 + 4t \Big|_{t=-3}^{t=2}$$

$$= \left[-\frac{2}{4}(16) - \frac{1}{3}(-8) + 8\right] - \left[-\frac{2}{4}(81) - \frac{1}{3}(-27) - 12\right]$$

$$= \left[\cancel{-8} - \frac{8}{3} + \cancel{8}\right] - \left[-\frac{81}{2} + 9 - 12\right]$$

$$= -\frac{8}{3} + \frac{81}{2} - 3 = -\frac{16}{6} + \frac{243}{6} - \frac{18}{6} = \frac{209}{6}$$

$$\begin{array}{r} {}^{2}27 \\ \times\,3 \\ \hline 81 \end{array} \qquad \begin{array}{r} 81 \\ \times\,3 \\ \hline 243 \end{array}$$

$$-\frac{16}{6} + \frac{243}{6}$$

$$\begin{array}{r} 3\,1\,13 \\ 24\cancel{3} \\ 16 \\ \hline 227 \end{array}$$

We saw that even though C_1 and C_2 have same start & end points,

$$\int_{C_1} \neq \int_{C_2}$$

Therefore this has "dependence on path".

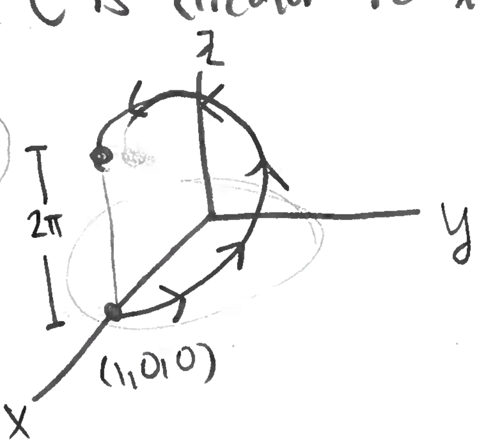
Once a parametrization is picked, it inherits an "orientation" (direction of travel).

Let "$-C$" denote C but w/ opposite orientation, then

$$\int_{-C} f\,ds = -\int_{C} f\,ds$$

Ex: Compute $\int_C y\sin(z)\,ds$ where C is circular helix given by

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = t \\ 0 \leq t \leq 2\pi \end{cases}$$



③

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}$$

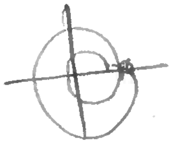
Therefore,

$$\int_C y\sin(z)\,ds = \int_0^{2\pi} \underbrace{\sin(t)\sin(t)}_{f(\vec{r}(t))} \underset{\overset{\uparrow}{\|\vec{r}\,'\|}}{\sqrt{2}}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \sin^2(t)\,dt$$

$\sin^2(u) = \frac{1-\cos(2u)}{2} \rightarrow$

$$= \sqrt{2}\left[\int_0^{2\pi} \frac{1}{2}\,dt - \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt\right]$$



$$= \sqrt{2}\left[\frac{1}{2}t\Big|_0^{2\pi} - \frac{1}{4}\cancel{\sin(2t)}\Big|_0^{2\pi}\right]$$

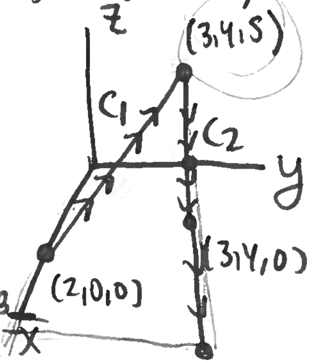
$$= \sqrt{2}\left[(\pi - 0) - \frac{1}{4}(0 - 0)\right]$$

$$= \sqrt{2}\,\pi$$

$$\left(\text{Note: } \int_C f \, \textcircled{dz} = \int_a^b f(\vec{r}(t)) \, z'(t) \, dt \right)$$

Ex: Compute $\int_C y\,dx + z\,dy + x\,dz$, where C is made up of

$$\begin{cases} C_1: \text{ line segment from } (2,0,0) \text{ to } (3,4,5) \\ C_2: \text{ line seg from } (3,4,5) \text{ to } (3,4,0) \end{cases}$$



$$\int_C = \int_{C_1} + \int_{C_2}$$

parametrize C_1

$$\begin{cases} \vec{r}_1(t) = t\langle 3,4,5\rangle + (1-t)\langle 2,0,0\rangle \\ \quad = \langle 3t, 4t, 5t\rangle + \langle 2-2t, 0, 0\rangle \\ \quad = \langle \underset{\uparrow\, x}{2+t}, \underset{\uparrow\, y}{4t}, \underset{\uparrow\, z}{5t}\rangle \\ 0 \le t \le 1 \end{cases}$$

$$\vec{r}_1{}'(t) = \langle \underset{\uparrow\, x'}{1}, \underset{\uparrow\, y'}{4}, \underset{\uparrow\, z'}{5}\rangle$$

parametrize C_2

$$\begin{cases} \vec{r}_2(t) = t\langle 3,4,0\rangle + (1-t)\langle 3,4,5\rangle \\ \quad = \langle 3t, 4t, 0\rangle + \langle 3-3t, 4-4t, 5-5t\rangle \\ \quad = \langle \overset{x}{3}, \overset{y}{4}, \overset{z}{5-5t}\rangle \\ 0 \le t \le 1 \end{cases}$$

$$\vec{r}_2{}'(t) = \langle \underset{\uparrow\, x'}{0}, \underset{\uparrow\, y'}{0}, \underset{\uparrow\, z'}{-5}\rangle$$

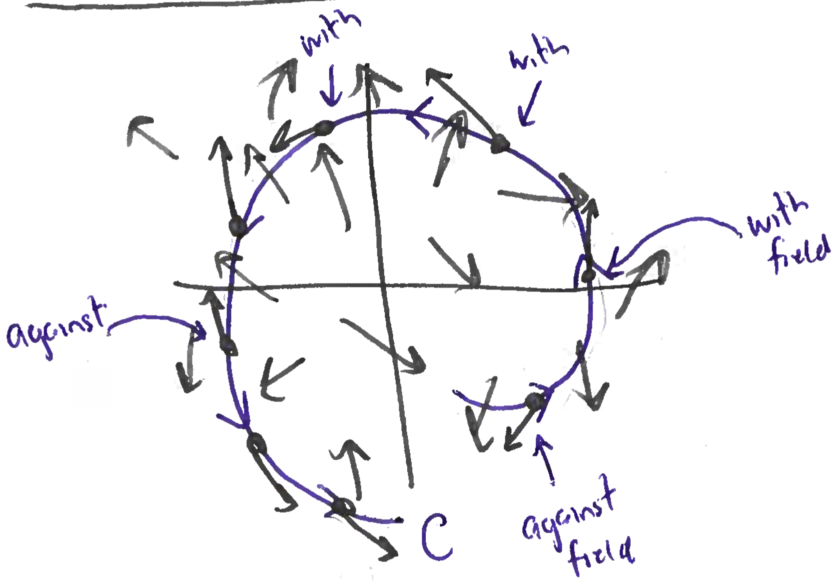
So, $$\int_C y\,dx + z\,dy + x\,dz = \left[\int_{C_1} + \int_{C_2}\right] y\,dx + z\,dy + x\,dz$$

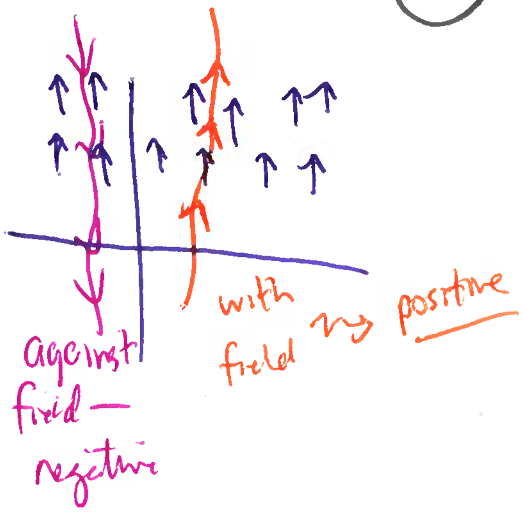
$$= \int_0^1 \underbrace{4t(1) + (5t)\overset{(4)}{(0)} + (2+t)5}_{4t + 20t + 5t + 10}\,dt$$

$$+ \int_0^1 0 + 0 + 3(-5)\,dt = \int_0^1 29t + 10\,dt - 15\int_0^1 1\,dt$$

$$= \frac{29}{2} + 10 - 15$$

Line integrals over vector fields





Def: If C is a curve given by $\begin{cases}\vec r(t) \\ a \le t \le b\end{cases}$, then the line integral of $\vec F$ over C is

$$\int_C \vec F \cdot d\vec r = \int_a^b \underbrace{\vec F(\vec r(t))}_{\text{vector}} \cdot \vec r\,'(t)\, dt$$

note: $\vec F = \langle P, Q, R \rangle$

$\vec r(t) = \langle x(t), y(t), z(t) \rangle$

$$\Rightarrow \int_C \vec F \cdot d\vec r = \int_C P\,dx + Q\,dy + R\,dz$$ ← what we just did!