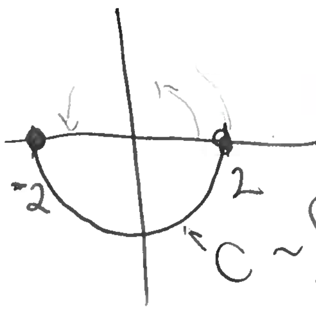
Ex: $\int_C 5+xy^2\,ds$ where C is lower half of circle $x^2+y^2=4$.

Soln:



$$C \sim \begin{cases} \vec{r}(t) = \langle \overset{x}{2\cos(t)}, \overset{y}{2\sin(t)} \rangle \\ \pi \le t \le 2\pi \end{cases}$$

$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t) \rangle$

$\|\vec{r}\,'\| = 2$

Calculate

$$\int_C 5+xy^2\,ds = \int_\pi^{2\pi} \left(5+(2\cos(t))(2\sin(t))^2\right) \|\vec{r}\,'(t)\|\,dt$$

$$= \int_\pi^{2\pi} \left(5+8\cos(t)\sin^2(t)\right)(2)\,dt$$

$$= 2\int_\pi^{2\pi} 5\,dt + 16\int_\pi^{2\pi} \cos(t)\sin^2(t)\,dt$$

$$= 10\pi + 16\int_0^0 u^2\,du$$

$$= 10\pi$$

$u=\sin(t)$

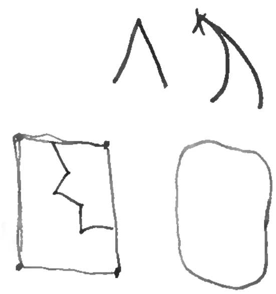
$du=\cos(t)dt$

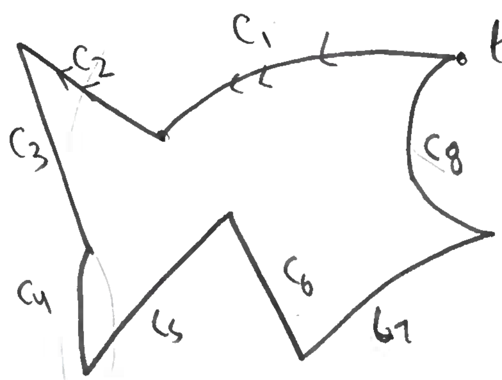
$t=\pi \rightarrow u=\sin(\pi)=0$

$t=2\pi \rightarrow u=\sin(2\pi)=0$

$\int_a^a = 0$

Suppose now C is "piecewise smooth"





← whole curve

$C = C_1 \cup C_2 \cup \ldots \cup C_8$

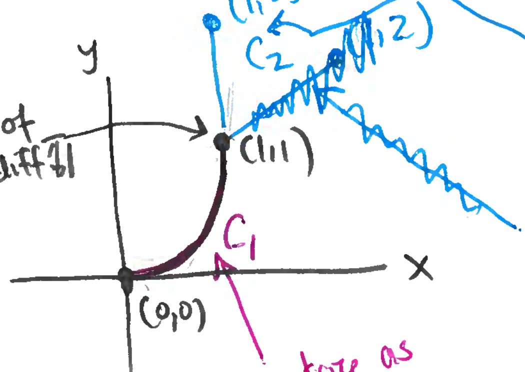
$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \ldots + \int_{C_8} f\,ds$$

Ex: Find $\int_C 2x\,ds$ where C is made of C_1 and C_2,

where $\begin{cases} C_1: \text{parabola } y=x^2 \text{ from } (0,0) \text{ to } (1,1) \\ C_2: \text{line segment from } (1,1) \text{ to } (1,2) \end{cases}$

could do
$r = \langle 1, t\rangle$
$1 \le t \le 2$

Draw:



parametrize as segment from $(1,1)$ to $(1,2)$

$\vec{r}_2(t) = t\langle 1,2\rangle + (1-t)\langle 1,1\rangle$

$0 \le t \le 1$ $= \langle t, 2t\rangle + \langle 1-t, 1-t\rangle$

$= \langle 1, 1+t\rangle$

in general
$y = f(x)$
parametrized as
$\langle t, f(t)\rangle$

parametrize as

$\begin{cases} \vec{r}_1(t) = \langle t, t^2\rangle \\ 0 \le t \le 1 \end{cases}$

$\vec{r}_1{}'(t) = \langle 1, 2t \rangle$

$\|\vec{r}_1{}'(t)\| = \sqrt{1+4t^2}$

$\vec{r}_2{}'(t) = \langle 0, 1 \rangle$

$\|\vec{r}\,'(t)\| = 1$

Compute

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \int_0^1 2t\sqrt{1+4t^2}\,dt + \int_0^1 2(1)(1)\,dt$$

$u = 1+4t^2$

$du = 8t\,dt$

$\frac{1}{8}du = t\,dt$

$t = 0 \to u = 1+0 = 1$

$t = 1 \to u = 1+4(1^2) = 5$

$$= \frac{2}{8}\int_1^5 u^{1/2}\,du + 2$$

$$= \frac{1}{4} \left. \frac{u^{3/2}}{3/2} \right|_{u=1}^{u=5} + 2$$

$$= \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)\left[5^{3/2} - 1\right] + 2$$

Ex: A wire shaped into a semicircle of rad 1 (with $y \geq 0$).

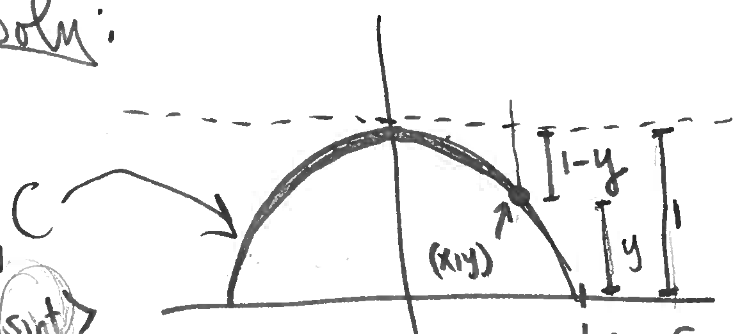
It is thicker at top: its linear density is proportional to distance from line $y=1$.

Find mass of wire.

$\rho(t)$ $\frac{\text{mass}}{\text{unit length}}$

$\int \rho(t)$

Soln:



$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \leq t \leq \pi$

$\vec{r}\,' = \langle -\sin(t), \cos(t) \rangle$

$\|\vec{r}\,'\| = 1$

So our density is of form

$$\boxed{\rho(x,y) = k(1-y)}$$

k: proportionality CONSTANT

$$\boxed{\text{mass} = \int_C \rho(x,y)\,ds}$$

Calculate

$$\text{mass} = \int_C \rho(x,y)\,ds = k\int_0^{\pi} (1-\sin(t))(1)\,dt$$

$$= k\pi - \left[-\cos(t)\right]_0^{\pi}$$

$$= k\pi - [-(-1) - (-1)]$$

$1+1$

$$= k\pi - 2$$

We know $\int_C \sim \, ds$

What about $\int_C \sim \, dx$ or $\int_C \sim \, dy$

Def: $\displaystyle\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$

$\vec{r}(t)$

$a \le t \le b$

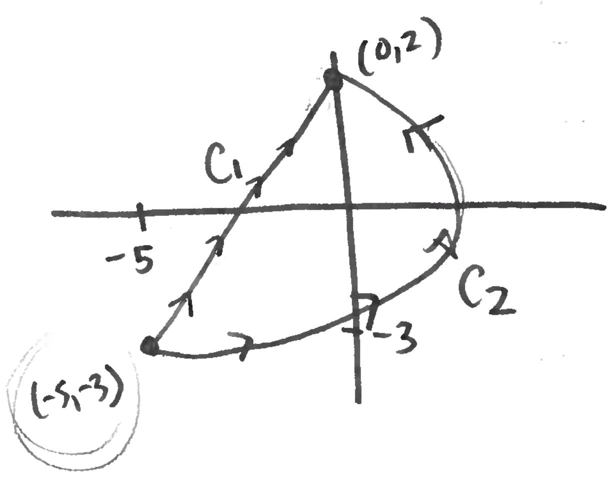
$\displaystyle\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$

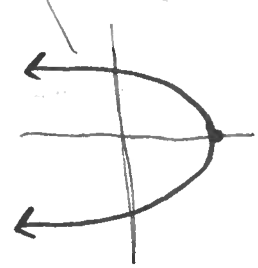
$x = y^2$ $\quad$ $x = -y^2$

Ex: Calculate $\displaystyle\int_C y^2\,dx + x\,dy$ where

(a) $C = C_1$ is line segment from $(-5,-3)$ to $(0,2)$

(b) $C = C_2$ is arc of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$





on C_1

$$\begin{cases} \vec{r}(t) = t\langle 0,2\rangle + (1-t)\langle -5,-3\rangle \\ \quad = \langle 0,2t\rangle + \langle -5+5t, -3+3t\rangle \\ \quad = \langle \underbrace{-5+5t}_{x}, \underbrace{-3+5t}_{y}\rangle \\ 0 \le t \le 1 \end{cases}$$

$$\vec{r}\,'(t) = \langle \underset{x'}{5}, \underset{y'}{5}\rangle$$

Calculate

$$\int_{C_1} y^2\,dx + x\,dy = \int_{C_1} y^2\,dx + \int_{C_1} x\,dy$$

$$= \int_0^1 (-3+5t)^2\,5\,dt + \int_0^1 (-5+5t)5\,dt$$

$$= 5\int_0^1 25t^2 - 30t + 9\,dt + 5\int_0^1 5t - 5\,dt$$

$$= 5\left[\frac{25}{3} - \frac{30}{2} + 9\right] + 5\left[\frac{5}{2} - 5\right]$$

$$= 5\left[\frac{50}{6} - \frac{90}{6} + \frac{54}{6}\right] + 5\left[\frac{15}{6} - \frac{30}{6}\right]$$

$$= 5\left[\frac{14}{6} - \frac{15}{6}\right] = -\frac{5}{6}$$

Mon: do $\int_{C_2}$