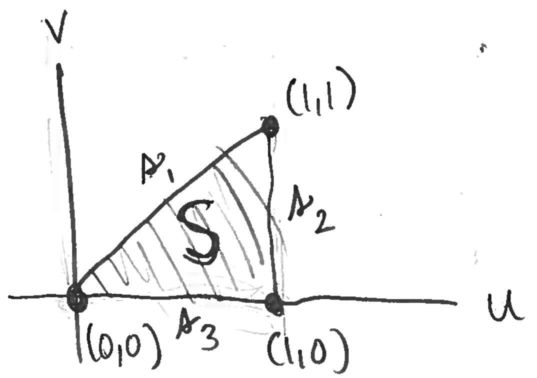
EX: Show what the transformation

$$\begin{cases} x = u \\ y = v^2 \end{cases}$$

does to triangle w/ vertices $(0,0)$, $(1,1)$, and $(1,0)$.

Soln:



s_1

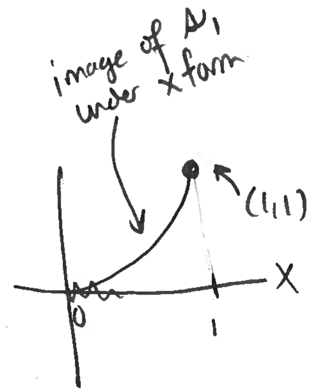
$u = v$, $0 \le u, v \le 1$

$$\begin{cases} x = u = v = \sqrt{y} \\ v = \pm\sqrt{y} \end{cases} \rightarrow \boxed{\begin{array}{l} 0 \le x \le 1 \\ x = \sqrt{y} \end{array}}$$

$y = x^2$

no negative soln b/c

$y = v^2$ never negative for $0 \le v \le 1$

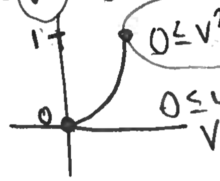


s_2

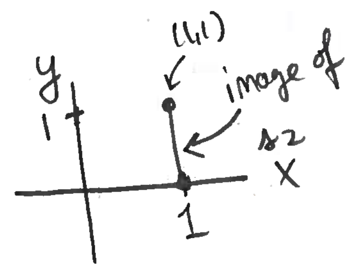
$u = 1$, $0 \le v \le 1$

$$\begin{cases} x = 1 \\ y = v^2 \end{cases} \quad 0 \le v \le 1$$

$0 \le v^2 \le 1$



$0 \le y \le 1$

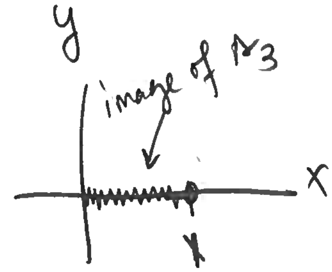


s_3

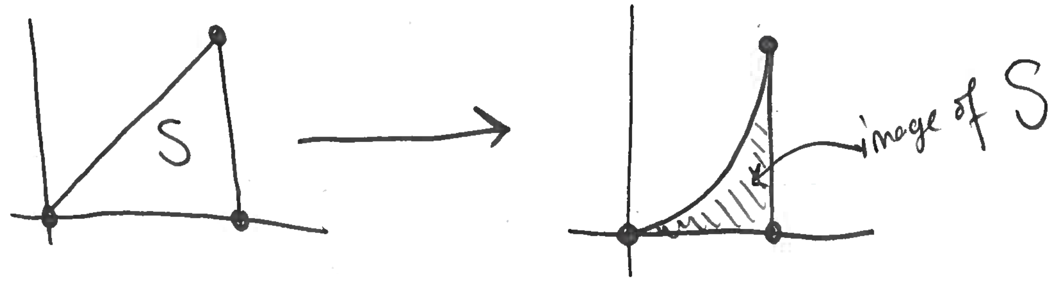
$v=0, \quad 0 \le u \le 1$

$\downarrow$

$$\begin{cases} x = u & 0 \le u \le 1 \\ & 0 \le x \le 1 \\ y = 0^2 = 0 & \end{cases}$$



Combining:



<u>Recall</u>: $\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

"Jacobian"

and

$$\iint f(x,y)\, dA = \iint f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

<u>Ex</u>: Evaluate $\iint_R e^{\frac{x+y}{x-y}}\,dA$ where R is the trapezoid region w/ vertices $(1,0),(2,0),(0,-2),(0,-1)$.

<u>Soln</u>: What change of var to use?

$$\begin{cases} u = x+y & (i) \\ v = x-y & (ii) \end{cases}$$

nice b/c $\iint$ become $\iint e^{\frac{u}{v}}\,du\,dv$
I can do this

However – this is not of correct form

$$\begin{cases} x = \text{\_\_\_} \\ y = \text{\_\_\_} \end{cases}$$

Eqt (i) $\rightarrow$ $x = u - y$

$\downarrow$ (ii)

$v = (u-y) - y \rightarrow v - u = -2y$

$\rightarrow y = \frac{1}{2}(u-v)$

$$\begin{aligned} x &= u - \left(\tfrac{1}{2}\right)(u-v) \\ &= u - \tfrac{1}{2}u + \tfrac{1}{2}v \\ &= \tfrac{1}{2}(u+v) \end{aligned}$$

$\Rightarrow$ use

$$\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

Compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

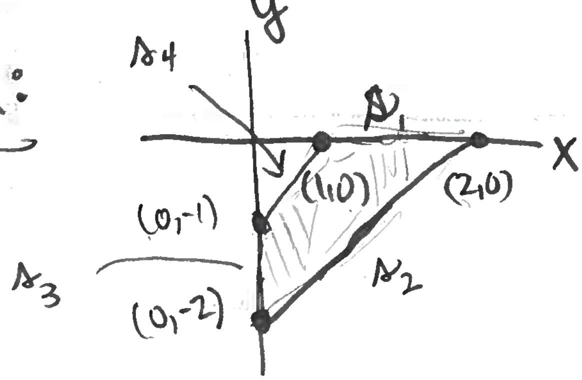
$$= \det\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

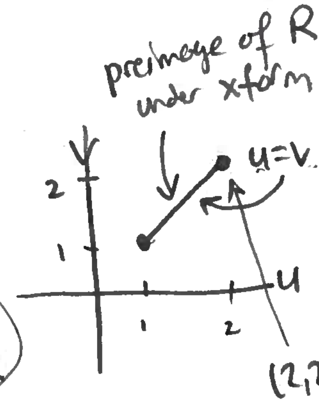
$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\Rightarrow \boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}}$$

We have to find the region S in uv-plane which maps to R.

Draw R:





$\underline{s_1}$

$y=0,\ 1 \le x \le 2 \rightarrow \begin{cases} u = x \\ v = x \end{cases} \rightarrow \begin{matrix} u = v \\ 1 \le u \le 2 \end{matrix}$

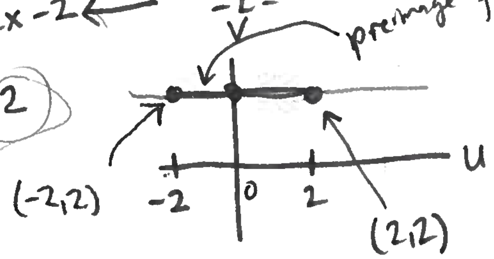
$\underline{s_2}$ $y = x-2,\ 0 \le x \le 2$

$\downarrow$

$\begin{cases} u = x + (x-2) = 2x-2 \\ v = x-(x-2) = 2 \end{cases}$

at $\frac{x=0}{u=-2}$ at $\frac{x=2}{u=2}$

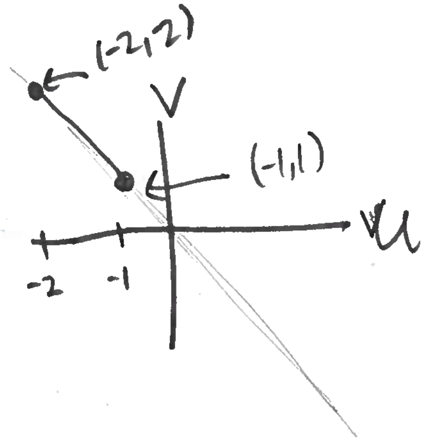
$-2 \le u \le 0$



A_3

$x=0, \quad -2 \le y \le -1$

$$\begin{cases} u = y \\ v = -y \end{cases} \quad u=-v, \quad -2 \le u \le -1$$



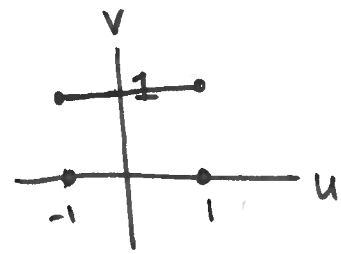
A_4

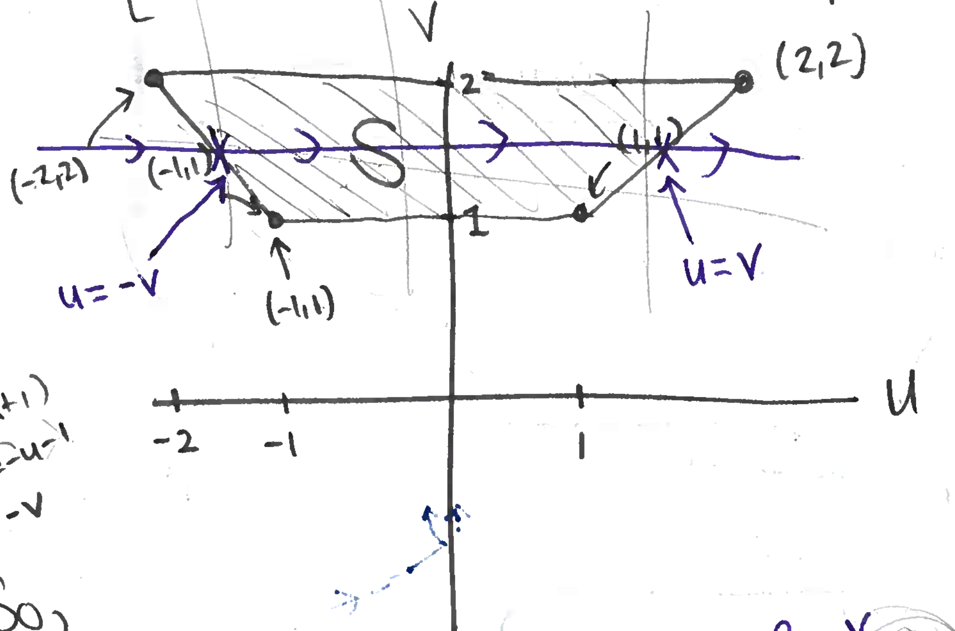
$y = x-1, \quad 0 \le x \le 1$

$$\begin{cases} u = x+(x-1) = 2x-1 \\ v = x-(x-1) = 1 \end{cases}$$

$\underline{x=0}$ $u=-1$ $\quad$ $\underline{x=1}$ $u=1$

$-1 \le u \le 1$





Slope $= \frac{1-2}{-1+2} = \frac{-1}{1}$

$v-1 = -(u+1)$

$v-1 = -u-1$

$u=-v$

$\int e^{\frac{x}{5}} = 5e^{\frac{x}{5}}$

So,

$$\iint_R e^{\frac{x+y}{x-y}} dA = \iint_S e^{\frac{u}{v}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA = \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} du\, dv$$

$$= \frac{1}{2}\int_1^2 ve^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv \quad \frac{v^2}{2}$$

$$= \frac{1}{2}\int_1^2 v[e^1 - e^{-1}]dv = \frac{e-\frac{1}{e}}{2}\int_1^2 v\, dv = \frac{e-\frac{1}{e}}{2}\left(2-\frac{1}{2}\right)$$

Triple ∫∫∫

Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

Spherical

"u" $= \rho$ "v" $= \phi$ "θ" $= w$

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$$

Here,

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det\begin{bmatrix} s(\phi)c(\theta) & \rho c(\phi)c(\theta) & -\rho s(\phi)s(\theta) \\ s(\phi)s(\theta) & \rho c(\phi)s(\theta) & \rho s(\phi)c(\theta) \\ c(\phi) & -\rho s(\phi) & 0 \end{bmatrix}$$

during lecture accidentally wrote 0 here

$$= s(\phi)c(\theta)\left(0 + \rho^2 s^2(\phi)\, c(\theta)\right)$$

$$- \rho c(\phi)c(\theta)\left(0 - \rho s(\phi)c(\theta)c(\phi)\right)$$

$$+ \left(-\rho s(\phi)s(\theta)\right)\underbrace{\left(-\rho s^2(\phi)s(\theta) - \rho c^2(\phi)s(\theta)\right)}_{-\rho s(\theta)\left(\underbrace{s^2(\phi)+c^2(\phi)}_{=1}\right)}$$

where's the mistake?

one of these

$$= \rho^2 s^{3}(\phi)\, c^2(\theta)$$

$$+ \rho^2 c^2(\phi) c^2(\theta) s(\phi)$$

$$+ \rho^2 s(\phi) s^2(\theta)$$

want

$$= \rho^2 s(\phi) c^2(\theta)\left[\underbrace{s^2(\phi) + c^2(\phi)}_{=1}\right] + \boxed{\rho^2 s(\phi)} s^2(\theta) = \rho^2 s(\phi)\left[\underbrace{c^2(\theta) + s^2(\theta)}_{=1}\right]$$

$$= \rho^2 s(\phi)$$