We have seen three different coord transforms

Double ints: polar coords

extra

$dy\,dx \mapsto r\,dr\,d\theta$

where does "extra" stuff come from?

$\iiint$: cylindrical

extra

$dy\,dx\,dz \mapsto r\,dr\,d\theta\,dz$

$\iiint$: spherical

extra

$dy\,dx\,dz \mapsto \rho^2 \sin(\phi)\,d\rho\,d\theta\,d\phi$

Change of variables in higher dimension

in one variable: "u-substitution"

Ex: $\int e^{x^2-1}(2x)\,dx$

Soln: $u = x^2-1$ $\Rightarrow \int 2xe^{x^2-1}\,dx = \int e^u\,du$

$du = 2x\,dx$

In general:

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{u=g^{-1}(a)}^{u=g^{-1}(b)} f(g(u))\,g'(u)\,du$$

"extra"

$x = g(u)$

$dx = g'(u)\,du$

$x=a \to a = g(u)$

$\downarrow$

$u = g^{-1}(a)$

(if exists)

$x=b \to u = g^{-1}(b)$

Def (Jacobian) for $\iint$: Consider

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

Then the determinant Jacobian ~~matrix~~ is

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

this is the "extra" stuff

In polar:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$= r\cos^2(\theta) - (-r\sin^2(\theta))$$

$$= r\left[\cos^2(\theta) + \sin^2(\theta)\right]$$

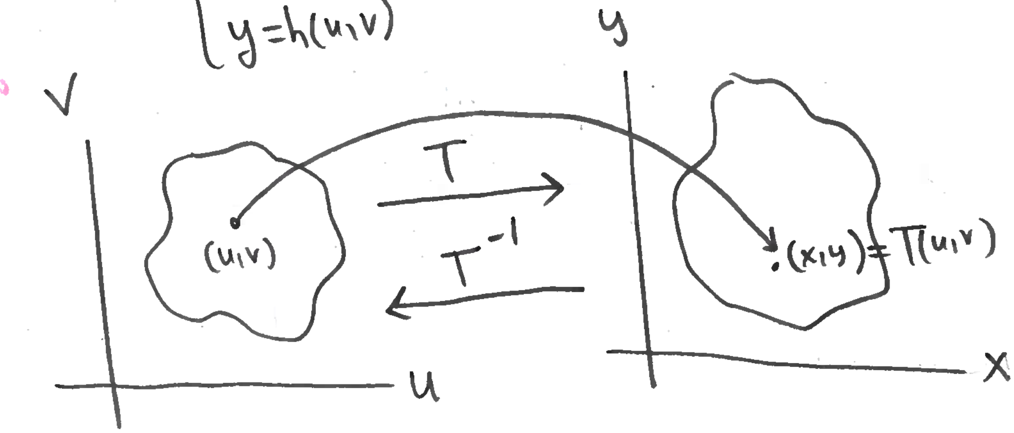
$$= r$$

What is the geometry of this?

Transformations from uv-plane to xy-plane:

$$T(u,v) = (x,y)$$

where $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$



T is just a function whose domain & range is in $\mathbb{R}^2$

Ex: Consider

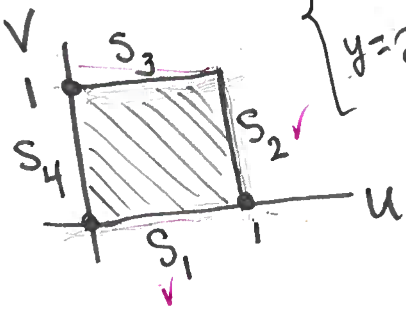
$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

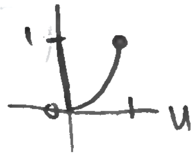
T[S] output of plugging all points of S into T

Find image of square $S = \left\{(u,v) : \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 1 \end{matrix}\right\}$ under the map T.

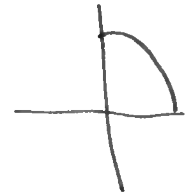
Draw S

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$





$y = 1 - x^2$

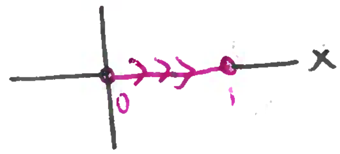


S_1

$v = 0,\ 0 \le u \le 1$ gives us

$$\begin{cases} x = u^2 \\ y = 0 \end{cases}$$

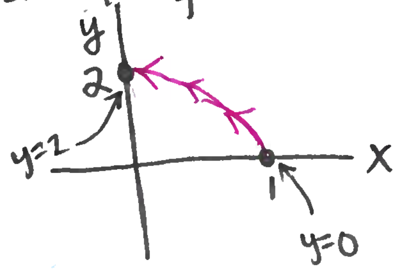
$0 \le u^2 \le 1 \Rightarrow \boxed{0 \le x \le 1}$



$\underline{S_2}$: $u = 1,\ 0 \le v \le 1$ gives us

$$\begin{cases} x = 1 - v^2 \rightarrow x = 1 - \left(\frac{y}{2}\right)^2 = 1 - \frac{y^2}{4} \\ y = 2v \rightarrow v = \frac{y}{2} \end{cases}$$

$0 \le y \le 2$

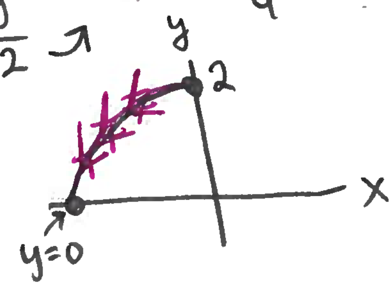


S_3: $v = 1,\ 0 \le u \le 1$

$$\begin{cases} x = u^2 - 1 \\ y = 2u \rightarrow u = \frac{y}{2} \end{cases}$$

$x = \frac{y^2}{4} - 1$

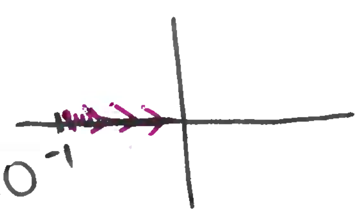
$0 \le y \le 2$



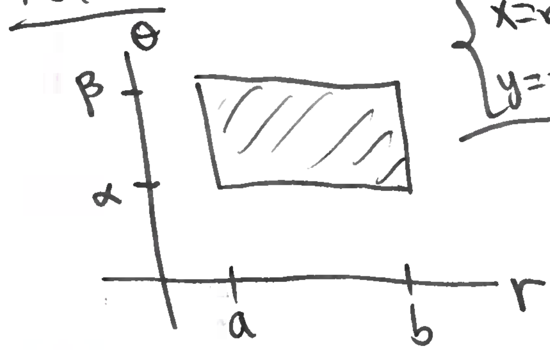
S_4: $u = 0,\ 0 \le v \le 1$

$$\begin{cases} x = -v^2 \\ y = 0 \end{cases}$$

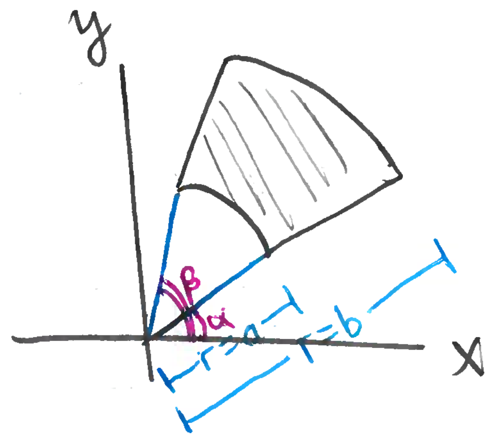
$-1 \le x \le 0$



Polar



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



Ex: Use c.o.v. $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$ to evaluate

$\iint_R y\,dA$ where R is region bdd by x-axis & parabolas $y^2 = 4-4x$ and $y^2 = 4+4x$; $y \geq 0$.

Calculate

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \det\begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = 4u^2 + 4v^2$$

So,

$$\iint_R y\,dA = \int_0^1\int_0^1 2uv\underbrace{(4u^2+4v^2)}_{\text{extra}}\,du\,dv$$

$$= 8\int_0^1\int_0^1 u^3v + uv^3\,du\,dv$$

$$= 8\int_0^1 \frac{u^4}{4}v + \frac{u^2}{2}v^3\Big|_{u=0}^{u=1}\,dv$$

$$= 8\int_0^1 \frac{v}{4} + \frac{v^3}{3}\,dv$$

$$= 8\left[\frac{v^2}{8} + \frac{v^4}{12}\right|_0^1$$

$$= 8\left(\frac{1}{8} + \frac{1}{12}\right)$$