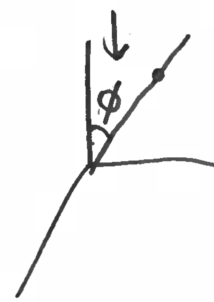
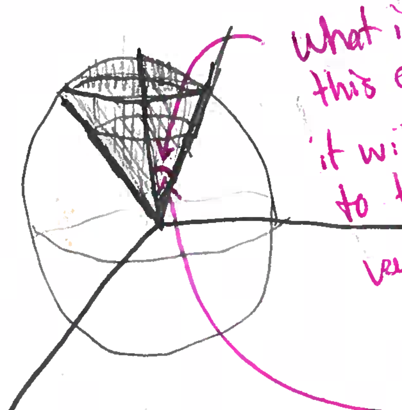
EX: Find vol of a solid lying above cone $z=\sqrt{x^2+y^2}$ + below sphere $x^2+y^2+z^2=1$.

"azimuthal angle"



Soln:



What is this angle? it will correspond to the largest value of ϕ

Where does cone intersect sphere?

Plug $z=\sqrt{x^2+y^2}$ into $x^2+y^2+z^2=1$

$$2x^2+2y^2=1$$

$$\boxed{x^2+y^2=1/2}$$

plug in $x=\rho\sin(\phi)\cos(\theta)$

$y=\rho\sin(\phi)\sin(\theta)$

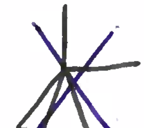
realize: $\rho=1$ (b/c we on sphere)

$$\sin^2(\phi)\cos^2(\theta)+\sin^2(\phi)\sin(\theta) = \frac{1}{2}$$

$$\sin^2(\phi)\underbrace{\left[\cos^2(\theta)+\sin^2(\theta)\right]}_{=1}=\frac{1}{2}$$

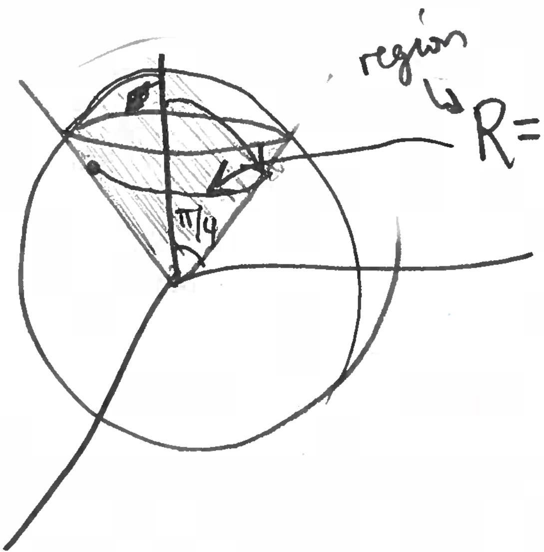
$$\sin^2(\phi)=\frac{1}{2} \rightarrow \sin(\phi)=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$$

(recall: $0\le\phi\le\pi$) $\rightarrow \phi=\frac{\pi}{4}, \cancel{\frac{3\pi}{4}}$ not our picture



②

Now:



region

$$R=\left\{(\rho,\theta,\phi):\ \begin{array}{l}0\le\rho\le 1\\ 0\le\theta\le 2\pi\\ 0\le\phi\le\frac{\pi}{4}\end{array}\right\}$$

So,

$$\mathrm{Vol}(R)=\iiint_R 1\,dV$$

extra

$$\frac{\rho^3}{3}\qquad =\int_0^{\pi/4}\int_0^{2\pi}\int_0^{1}\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

$$\int\sin\phi=-\cos(\phi)+C\qquad =\int_0^{\pi/4}\sin(\phi)\int_0^{2\pi}\left(\tfrac{1}{3}-0\right)d\theta\,d\phi$$

$$=\frac{1}{3}2\pi\int_0^{\pi/4}\sin(\phi)\,d\phi$$

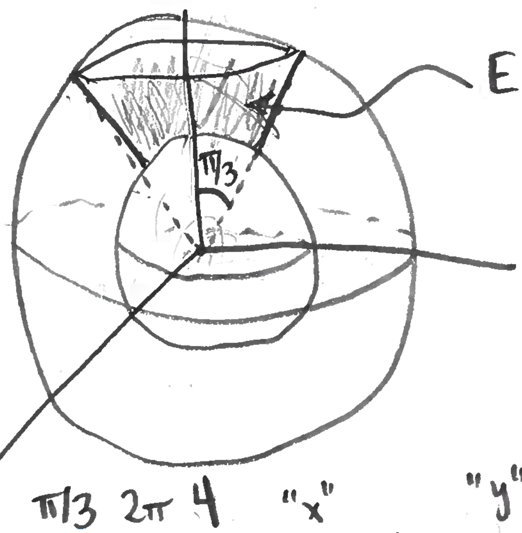
$$=\frac{1}{3}2\pi\left[-\cos\left(\tfrac{\pi}{4}\right)-\left(-\cos(0)\right)\right]$$

$$=\frac{2\pi}{3}\left[-\frac{\sqrt{2}}{2}+1\right]$$

Ex: $\iiint_E xyz\, dV$ where E is between spheres

$\rho = 2$, $\rho = 4$, + above cone $\phi = \pi/3$.

Soln:



$E = \left\{ (\rho, \theta, \phi) : \begin{array}{l} 2 \le \rho \le 4 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/3 \end{array} \right\}$

$$\iiint_E xyz\, dV = \int_0^{\pi/3}\int_0^{2\pi}\int_2^4 \underbrace{(\rho s(\phi) c(\theta))}_{\text{"x"}}\underbrace{(\rho s(\phi) s(\theta))}_{\text{"y"}}\underbrace{(\rho c(\phi))}_{\text{"z"}}\underbrace{(\rho^2 s(\phi))}_{\text{extra}}\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/3} \sin^3(\phi)\cos(\phi) \int_{\theta=0}^{\theta=2\pi} \cos(\theta)\sin(\theta) \int_2^4 \rho^5\, d\rho\, d\theta\, d\phi$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$\theta = 0 \rightarrow u = \sin(0) = 0$

$\theta = 2\pi \rightarrow u = \sin(2\pi) = 0$

$$= \left(\frac{4^6}{6} - \frac{2^6}{6}\right) \int_0^{\pi/3} \sin^3(\phi)\cos(\phi) \boxed{\int_0^0 u\, du}\, d\phi$$

$\int_0^0 u\, du = 0$

$= 0$

Ex: Calculate $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2 \, dz \, dx \, dy$

$x=\sqrt{9-y^2}$

$x^2+y^2=9 \leftarrow$ circle rad 3

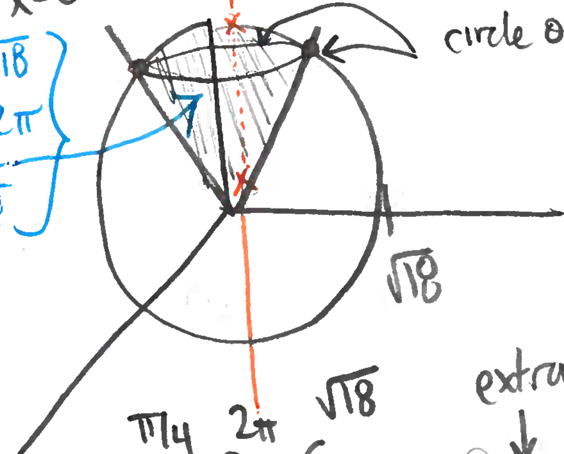
$z=\sqrt{18-x^2-y^2} \leftarrow x^2+y^2+z^2=18 \leftarrow$ sphere of radius $\sqrt{18}$ $\leftarrow \rho=\sqrt{18}$

$z=\sqrt{x^2+y^2} \leftarrow$ earlier: cone $\phi=\pi/4$

numbers   curves   surfaces

$x=0$

$R=\left\{(\rho,\theta,\phi):\ 0\le\rho\le\sqrt{18},\ 0\le\theta\le2\pi,\ 0\le\phi\le\frac{\pi}{4}\right\}$



circle of ∩ of cone + sphere

$\sqrt{18-x^2-y^2}=\sqrt{x^2+y^2}$

$\downarrow$

$18=2(x^2+y^2)$

$9=x^2+y^2$

extra

$$\text{Original} \iiint = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sqrt{18}} \rho^2 \left(\rho^2 \sin\phi\right) d\rho \, d\theta \, d\phi$$

$\frac{\rho^5}{5}$

$$=\int_0^{\pi/4} \sin(\phi) \int_0^{2\pi} \frac{(\sqrt{18})^5}{5} \, d\theta \, d\phi$$

$$=\frac{(\sqrt{18})^5}{5}(2\pi)\left[-\cos(\phi)\right]_0^{\pi/4}$$

$$=\frac{2\pi}{5}(\sqrt{18})^5\left[-\frac{\sqrt{2}}{2}+1\right]$$

Perspective

CALC 1,2: $f: \mathbb{R} \to \mathbb{R}$

CALC 3:

* $f: \mathbb{R} \to \mathbb{R}^n$ (space curves)

$\frac{\partial}{\partial x}, \iiint, \nabla \to$ * $f: \mathbb{R}^n \to \mathbb{R}$ (right now)

future * $f: \mathbb{R}^n \to \mathbb{R}^m$ (vector calc)

we go soon!