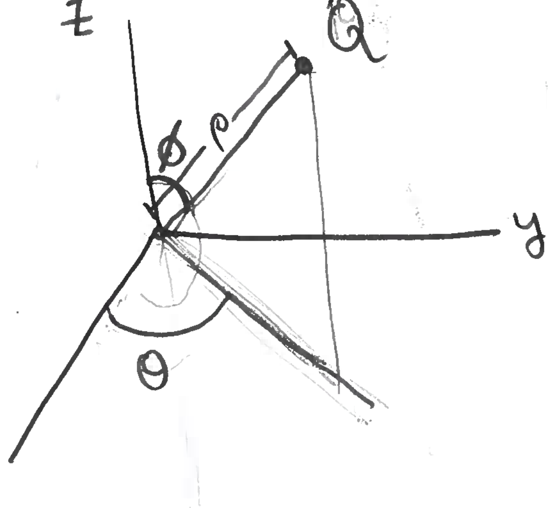
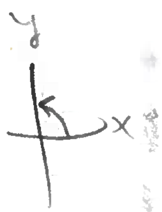
Spherical → much more involved than cylindrical

Def: The spherical coords $(\rho, \theta, \phi)$ of the point Q is shown:

("rho" — Greek r)



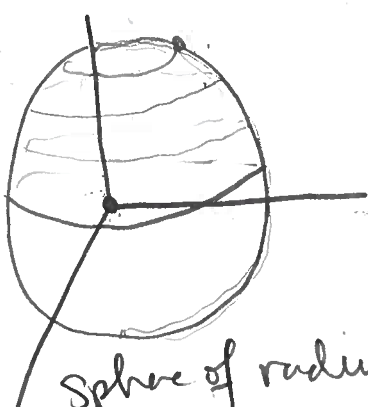




- $\rho$ - distance of Q from $(0,0,0)$
- $\theta$ — same as before — spins on xy-plane
- $\phi$ - angle b/w ~~+~~ -z axis + line segment from origin to Q

Always: $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

Some graphs

$\boxed{\rho = C}$ ← (constant)

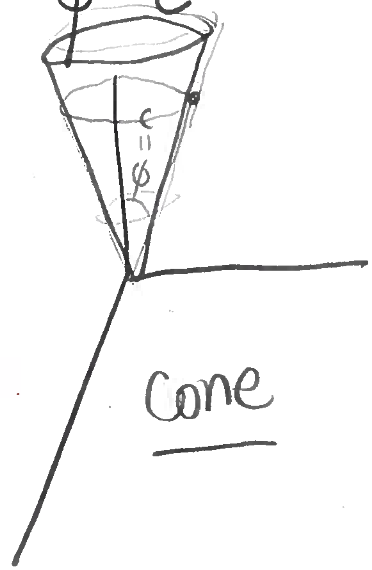


sphere of radius C

$\theta = c$


plane (half)

$\phi = c$




cone

②

"s" sin
"c" cos

Turns out:

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) = \rho\, s(\phi)c(\theta) \\ y = \rho\sin(\phi)\sin(\theta) = \rho\, s(\phi)s(\theta) \\ z = \rho\cos(\phi) = \rho\, c(\phi) \end{cases}$$

Can show:

$$x^2+y^2+z^2 = \rho^2 s^2(\phi)c^2(\theta) + \rho^2 s^2(\phi)s^2(\theta) + \rho^2 c^2(\phi)$$

$$= \rho^2 s^2(\phi)\underbrace{\left[c^2(\theta)+s^2(\theta)\right]}_{=1} + \rho^2 c^2(\phi)$$

$$= \rho^2 \underbrace{\left[s^2(\phi) + c^2(\phi)\right]}_{=1}$$

$$= \rho^2$$

## Integration

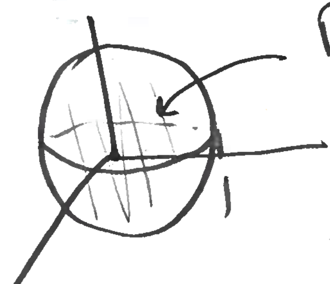
"extra stuff"

$$dx\,dy\,dz \longmapsto (\rho^2\sin(\phi))\,d\rho\, d\theta\, d\phi$$

<u>Ex</u>: Evaluate $\iiint_B e^{\left[(x^2+y^2+z^2)^{3/2}\right]}\,dV$

where B is ball $x^2+y^2+z^2 \le 1$.

<u>Soln</u>:



$B = \left\{ (\rho, \theta, \phi) :\ \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{array} \right\}$

Calculate

$$\iiint_{B} \exp\left((x^2+y^2+z^2)^{3/2}\right) dV$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_{\rho=0}^{\rho=1} \exp\Big(\underbrace{(\rho^2)^{3/2}}_{\rho^3}\Big) \overbrace{\rho^2 \sin(\phi)}^{\text{extra}}\, d\rho\, d\theta\, d\phi$$

$u = \rho^3$

$du = 3\rho^2\,d\rho$

$\frac{1}{3}du = \rho^2\,d\rho$

$\rho = 0 \to u = 0^3 = 0$

$\rho = 1 \to u = 1^3 = 1$

$$= \frac{1}{3}\int_0^{\pi} \sin(\phi) \int_0^{2\pi}\int_0^{1} e^{u}\, du\, d\theta\, d\phi$$

$$= \frac{e-1}{3}\int_0^{\pi} \sin(\phi) \underbrace{\int_0^{2\pi} 1\, d\theta}_{=2\pi}\, d\phi = \frac{2\pi}{3}(e-1)\int_0^{\pi} \sin(\phi)\, d\phi$$

antiderv $\hookrightarrow -\cos(\phi)$

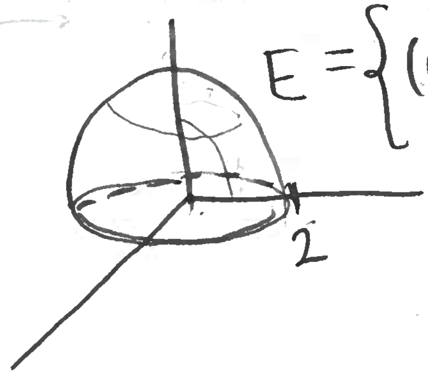
$$= \frac{2\pi}{3}(e-1)\Big[\underbrace{-\cos(\pi) - (-\cos(0))}_{1+1}\Big]$$

$$= \frac{4\pi}{3}(e-1)$$

④

Ex: $\iiint_E 16z\,dV$ where $E$ is interior of upper half of sphere $x^2+y^2+z^2=2$.

Soln:




$$E=\left\{(\rho,\theta,\phi):\ \begin{matrix}0\le\rho\le2\\0\le\theta\le2\pi\\0\le\phi\le\frac{\pi}{2}\end{matrix}\right\}$$

Calculate

$$\iiint_E 16z\,dV = 16\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2}\underbrace{[\rho\cos(\phi)]}_{\text{"z"}}\underbrace{(\rho^2\sin(\phi))}_{\text{extra}}\,d\rho\, d\theta\, d\phi$$

$$=16\int_0^{\pi/2}\cos(\phi)\sin(\phi)\int_0^{2\pi}\boxed{\int_0^{2}\rho^3\,d\rho}\,d\theta\, d\phi$$

$$\left.\frac{\rho^4}{4}\right|_0^2=\frac{16}{4}-0=4$$

$$=64\int_0^{\pi/2}\cos(\phi)\sin(\phi)\underbrace{\boxed{\int_0^{2\pi}1\,d\theta}}_{=2\pi}\,d\phi$$

$u=\sin\phi$

$du=\cos\phi\,d\phi$

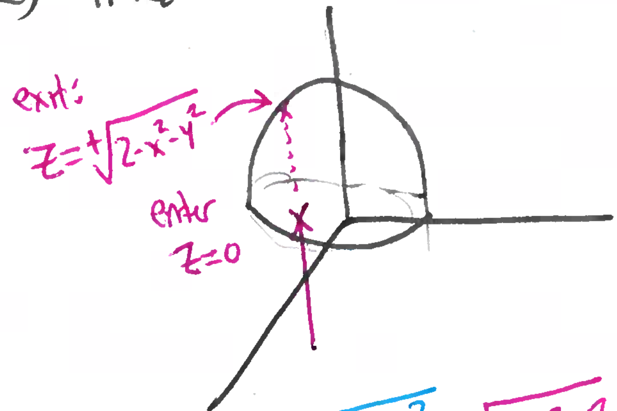
$\phi=0\rightarrow u=\sin(0)=0$

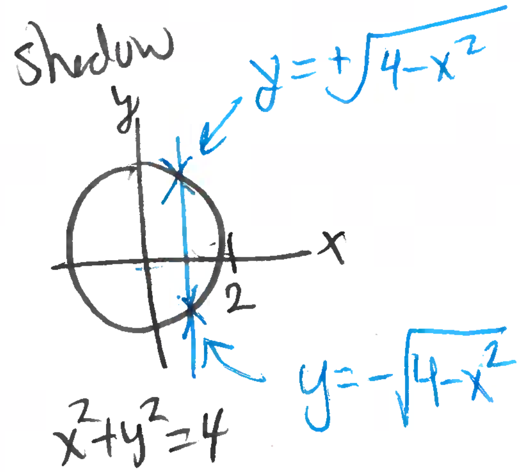
$\phi=\frac{\pi}{2}\rightarrow u=\sin(\frac{\pi}{2})=1$

$$=128\pi\int_{\phi=0}^{\phi=\pi/2}\cos(\phi)\sin(\phi)\,d\phi$$

$$=128\pi\underbrace{\int_0^1 u\,du}_{=1/2}=64\pi$$

Previous problem w/ no coord transform looks like: dz dy dx




$$\iiint_E 16z\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{2-x^2-y^2}} 16z\,dz\,dy\,dx$$