Center of mass of a flat object

Imagine object in plane

D

This object has density $\rho(x,y)$

$$\text{mass} = \iint_D \rho(x,y)\,dA$$

CALC 1 units

$f(x) \sim foo$

bar

$f'(x) \sim \frac{foo}{bar}$

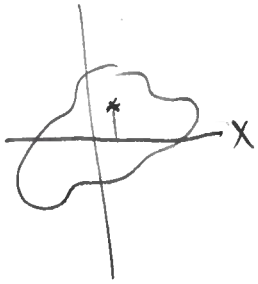
$\int f(x) \sim foo \cdot bar$

$\frac{\text{mass}}{\text{area}} = \frac{\text{mass}}{\text{length}^2}$

Moments

"moment about x-axis"

$$M_x = \iint_D y\,\rho(x,y)\,dA$$



"moment about y-axis"

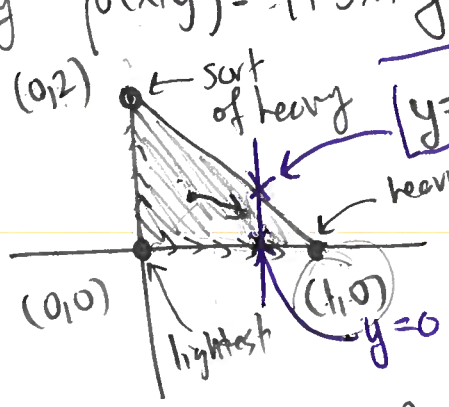
$$M_y = \iint_D x\,\rho(x,y)\,dA$$

Center of mass: $(\hat{x}, \hat{y})$,

where $\hat{x} = \frac{M_x}{m}$ and $\hat{y} = \frac{M_y}{m}$

Ex: Find center of mass of triangle w/ vertices $(0,0), (1,0), (0,2)$ & density $\rho(x,y) = 1+3x+y$.

Soln:



$$\text{mass} = m = \iint_D 1+3x+y\, dA$$

$$= \int_0^1 \int_0^{-2x+2} 1+3x+y\, dy\, dx$$

$$= \int_0^1 y+3xy+\frac{y^2}{2}\Big|_0^{-2x+2} dx \qquad \frac{1}{2}(4x^2-8x+4)$$

$$= \int_0^1 (-2x+2)+3x(-2x+2)+\frac{1}{2}(-2x+2)^2\, dx$$

$$= \int_0^1 -2x+2-6x^2+6x+2x^2-4x+2\, dx$$

$$= \int_0^1 -4x^2+4\, dx$$

$$= -\frac{4}{3}x^3+4x\Big|_0^1 = -\frac{4}{3}+4 = \frac{8}{3}$$

$$M_x = \iint_D y\rho(x,y)\,dA$$

$$= \int_0^1 \int_0^{-2x+2} y + 3xy + y^2 \, dy\,dx = \frac{11}{6}$$

$$M_y = \iint_D x\rho(x,y)\,dA$$

$$= \int_0^1 \int_0^{-2x+2} x + 3x^2 + yx \, dy\,dx = 1$$

So, center of mass is

$$(\hat{x}, \hat{y}) = \left(\frac{11/6}{8/3}, \frac{1}{8/3} \right)$$

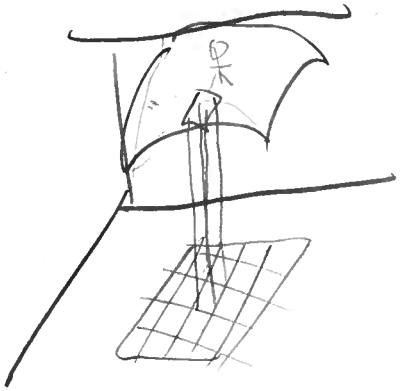
$$= \left(\frac{11}{16}, \frac{3}{8} \right)$$

$\frac{11}{6} \cdot \frac{3}{8}$

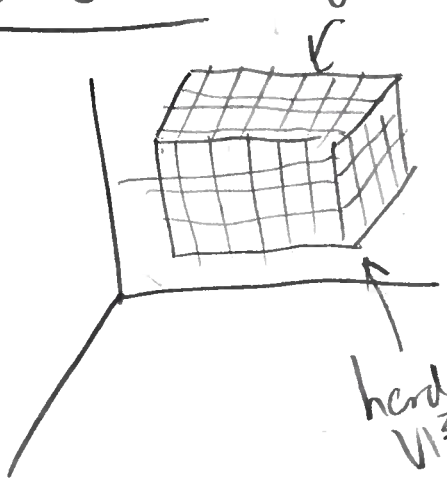
Triple integrals

CALC I

CALC 3 ∫∫



NOW ∫∫∫



ex) $f(x,y,z)$ ~ temp at (x,y,z) in a room

W/ ∫∫∫ ~ you get 6 ways to integrate in rect. coords:

$dy\,dx\,dz$ $dx\,dy\,dz$ $dz\,dx\,dy$

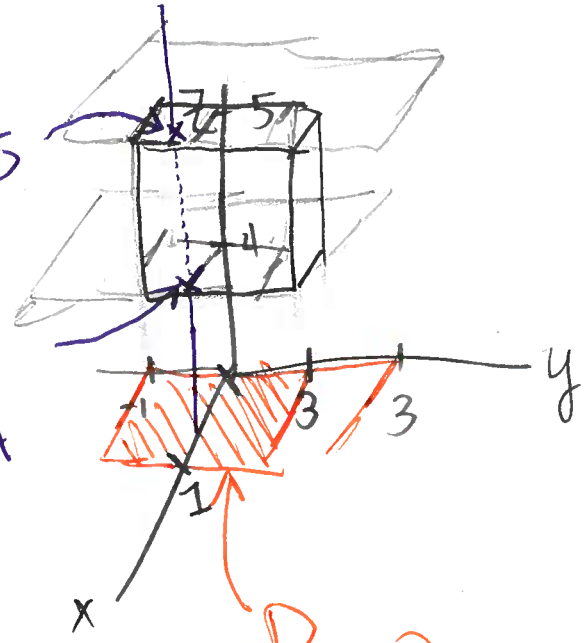
$dy\,dz\,dx$ $dx\,dz\,dy$ $dz\,dy\,dx$

Notation

$$\int_{\#}^{\#} dx$$

$$\int_{\#}^{\#}\int_{\text{curve}}^{\text{curve}} dA$$

$$\int_{\#}^{\#}\int_{\text{curve}}^{\text{curve}}\int_{\text{surface}}^{\text{surface}} dV$$



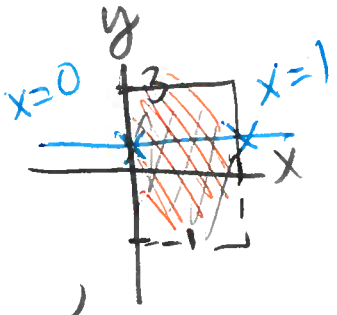
Ex: Evaluate

$$\iiint_B xyz^2\,dV$$

where $B = \{(x,y,z): 0\le x\le 1,\ -1\le y\le 3,\ 4\le z\le 5\}$

$$\iiint_B xyz^2\,dV = \iint_D \left(\int_4^5 xyz^2\,dz\right) dA$$

$\int xyz^2\,dz = \frac{xyz^3}{3}$

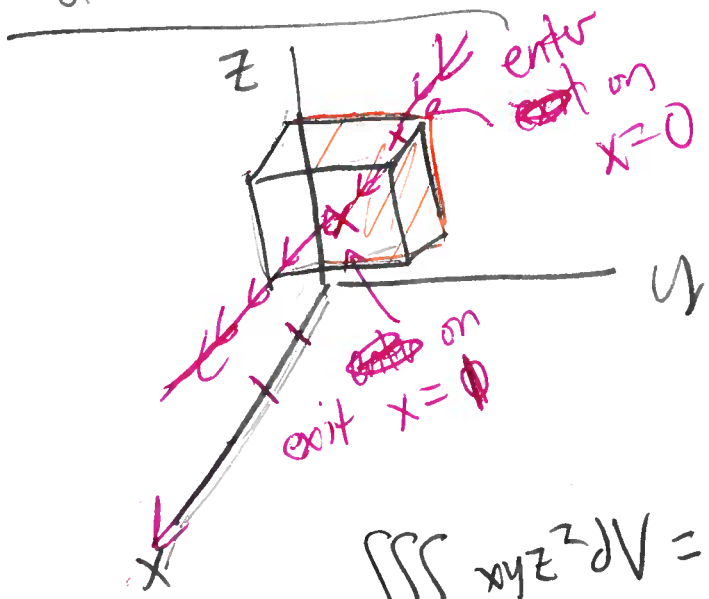


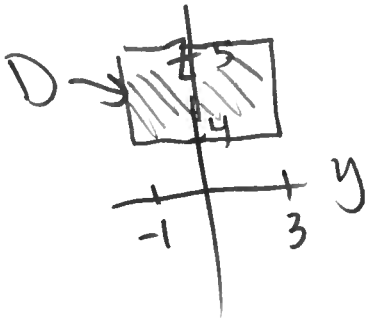
$$= \left(\frac{5^3}{3} - \frac{4^3}{3}\right)\iint_D xy\,dA$$

$$= \left(\frac{5^3}{3} - \frac{4^3}{3}\right)\int_{-1}^{3}\int_{0}^{1} xy\,dx\,dy$$

6

do ws dx dA





$$\iiint_B xyz^2\,dV = \iint_D \int_0^{1} xyz^2\,dx\,dA$$