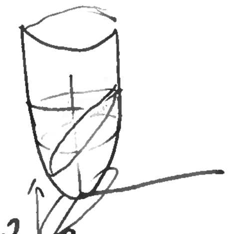
Lagrange multipliers

EX: Find max + min values of

$$\begin{cases} f(x,y) = 81x^2 + y^2 \\ \underbrace{4x^2 + y^2}_{g(x,y)} = \underbrace{9}_{K} \end{cases}$$



x^2/[illegible] $+ \frac{y^2}{4} = \frac{9}{4}$

$x = \frac{3}{2}\cos(t)$

$y = 3\sin(t)$

$z = 81x^2 + y^2 = 81\left(\frac{3}{2}\right)^2\cos^2(t) + (3)^2\sin^2(t)$

Soln: $\nabla f = \langle 162x, 2y \rangle$

$\nabla g = \langle 8x, 2y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = K \end{cases} \rightarrow \begin{cases} \langle 162x, 2y \rangle = \lambda \langle 8x, 2y \rangle \\ 4x^2 + y^2 = 9 \end{cases}$$

$x(162 - 8\lambda) = 0$

$$\rightarrow \begin{cases} 162x = 8\lambda x \rightarrow \lambda = \frac{162}{8} \text{ OR } x = 0 \\ 2y = 2\lambda y \rightarrow \lambda = 1 \text{ OR } y = 0 \\ 4x^2 + y^2 = 9 \quad \text{(iii)} \end{cases}$$

$\Downarrow$

WAYS IT CAN WORK

useless $\leftarrow 0 = 9$ FALSE $\xleftarrow{\text{(iii)}}$ * $x = 0,\ y = 0$

$\left(\frac{3}{2}, 0\right)$ $\left(-\frac{3}{2}, 0\right)$ $\leftarrow x = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2} \leftarrow 4x^2 + 0^2 = 9 \xleftarrow{\text{(iii)}}$ * $\lambda = \frac{162}{8},\ y = 0$

$(0,3)$ $(0,-3)$ $\leftarrow y = \pm 3 \leftarrow 0 + y^2 = 9 \xleftarrow{\text{(iii)}}$ * $\lambda = 1,\ x = 0$

(x,y)	$f(x,y)$
$(\frac{3}{2},0)$	182.85
$(-\frac{3}{2},0)$	182.85
$(0,3)$	9
$(0,-3)$	9

max is 182.85 + occurs at $(\frac{3}{2},0)$ and $(-\frac{3}{2},0)$

min is 9 + occurs at $(0,\pm 3)$

Ex: $\begin{cases} f(x,y,z) = xyz \\ \underbrace{x+9y^2+z^2}_{g} = \underset{\uparrow\atop K}{4} \end{cases}$ $\begin{cases} \nabla f = \lambda \nabla g \\ g = K \end{cases}$

$$\begin{cases} \langle yz, xz, xy \rangle = \lambda \langle 1, 18y, 2z \rangle \\ x+9y^2+z^2 = 4 \end{cases}$$

$$\begin{cases} \boxed{yz = \lambda} & \text{(i)} \\ xz = 18\lambda y & \text{(ii)} \\ xy = 2\lambda z & \text{(iii)} \\ x+9y^2+z^2 = 4 & \text{(iv)} \end{cases}$$

Plug (i) into (ii) + (iii) to get

$$\begin{cases} xz = 18y^2z & (v) \\ xy = 2yz^2 & (vi) \\ x + 9y^2 + z^2 = 4 & (vii) \end{cases}$$

Solve (v) for y^2z:

$$y^2z = \frac{xz}{18}$$

$$\downarrow (vi)$$

$$xy = 2\left(\frac{xz}{18}\right) = \frac{xz}{9}$$

$$xy - \frac{xz}{9} = 0$$

$$x\left(y - \frac{z}{9}\right) = 0$$

$x = 0$ OR $\boxed{y = \frac{z}{9}}$ (★)

$x = 0$:

$\downarrow$ (ii)

$y = 0$ OR $z = 0$

$y = 0 \xrightarrow{(iv)} z^2 = 4$, $z = \pm 2$

$z = 0 \xrightarrow{(iv)} y^2 = \frac{9}{4}$, $y = \pm \frac{3}{2}$

$y = \frac{28}{7}$

$y = \frac{z}{9}$:

$\downarrow$ (i): $\cancel{\frac{z^2}{9} = \lambda}$

$\downarrow$ (iv):

$$x + 9\left(\frac{z}{9}\right)^2 + z^2 = 4$$

$$\boxed{x = 4 - \frac{z^2}{9} - z^2 = 4 - \frac{10z^2}{9}} \quad (★★)$$

$\uparrow$ (vi)

(★) →

$$\frac{z}{9}\left(4 - \frac{10z^2}{9}\right) = 2\left(\frac{z}{9}\right)z^2$$

$$\frac{4z}{9} - \frac{10z^3}{81} = \frac{2z^3}{9}$$

$$\frac{4z}{9} - \frac{10z^3}{81} - \frac{18z^3}{81} = 0$$

$$\frac{4z}{9} - \frac{28z^3}{81} = 0$$

$\uparrow$ mult by 81

$$36z - 28z^3 = 0$$

$$4z(9 - 7z^2) = 0$$

$z = 0$ or $z = \pm \frac{3}{\sqrt{7}}$

FROM TOP:

	$z = 0$	$z = \frac{3}{\sqrt{7}}$	$z = -\frac{3}{\sqrt{7}}$
(★):	$y = 0$	$y = \frac{1}{3\sqrt{7}}$	$y = -\frac{1}{3\sqrt{7}}$
(★★):	$x = 4$	$x = 4 - \frac{10}{9}\left(\frac{9}{7}\right) = \frac{18}{7}$	$x = 4 - \frac{10}{9}\left(\frac{9}{7}\right) = \frac{18}{7}$

Check pts we found

(x,y,z)	$f(x,y,z)$
$(0,0,2)$	0
$(0,0,-2)$	0
$(0,\frac{3}{2},0)$	0
$(0,-\frac{3}{2},0)$	0
$(4,0,0)$	0
$(\frac{18}{7},\frac{1}{3\sqrt{7}},\frac{3}{\sqrt{7}})$	$\frac{18}{49}$
$(\frac{18}{7},-\frac{1}{3\sqrt{7}},-\frac{3}{\sqrt{7}})$	$\frac{18}{49}$

min of 0 at all these pts

max of $\frac{18}{49}$ at these