Def: A point (a,b) in plane is called a critical point of $f(x,y)$ if

$\sqrt{x} \xrightarrow{\frac{d}{dx}} \frac{1}{2\sqrt{x}}$

$$\boxed{f_x(a,b)=0 \text{ AND } f_y(a,b)=0}$$

OR

one of those partial derivatives doesn't exist

Theorem (2nd deriv test)

Spz all 2^{nd} partials of f are continuous near (a,b).
also suppose $f_x(a,b) = 0 = f_y(a,b)$.
Define

often the same

$$D = \det \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

$$= f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)f_{yx}(a,b)$$

Now:

(a) if $D>0$ and $f_{xx}(a,b)>0 \Rightarrow$ local min ~~max~~ at (a,b)

(b) if $D>0$ and $f_{xx}(a,b)<0 \Rightarrow$ local max at (a,b)

(c) if $D<0$, then a saddle point (neither local max nor local min)

(d) $D=0$, then no conclusion can be made using this test

Ex: Find & classify all local extrema and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

Soln: 1st Find cp's

$$\begin{cases} f_x = 4x^3 - 4y \overset{\text{set}}{=} 0 & \text{(i)} \\ f_y = 4y^3 - 4x \overset{\text{set}}{=} 0 & \text{(ii)} \end{cases}$$

From (i) $\rightarrow$ $\boxed{y = x^3}$

plug into (ii)

$$4(x^3)^3 - 4x = 0$$

$$4x^9 - 4x = 0$$

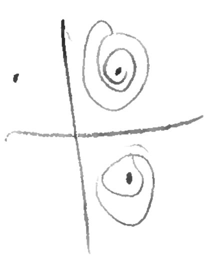
$$x(x^8 - 1) = 0$$

$x = 0$ OR $x^8 - 1 = 0$

$x^8 = 1$

$x = \pm \sqrt[8]{1} = \pm 1$

$x = 1$ $\quad$ $x = -1$



$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$ is a cp

$x = 1 \Rightarrow y = 1^3 = 1 \Rightarrow (1,1)$ is a cp

$x = -1 \Rightarrow y = (-1)^3 = -1 \Rightarrow (-1,-1)$ is a cp

Compute all 2^{nd} derivs:

$f_{xx} = 12x^2$ $\quad$ $f_{xy} = -4$ $\quad \Rightarrow D = f_{xx} f_{yy} - f_{xy} f_{yx}$

$f_{yx} = -4$ $\quad$ $f_{yy} = 12y^2$ $\quad = 144x^2y^2 - 16$

CP's

$(0,0), (1,1), (-1,-1)$

at $(0,0)$: $D(0,0) = 144 \cdot 0^2 \cdot 0^2 - 16 = -16 < 0$

$\Downarrow$

saddle pt at $(0,0)$

at $(1,1)$: $D(1,1) = 144 \cdot 1^2 \cdot 1^2 - 16 > 0$

$f_{xx}(1,1) = 12 \cdot 1^2 = 12 > 0$

$\Downarrow$

local min at $(1,1)$

at $(-1,-1)$: $D(-1,-1) = 144(-1)^2(-1)^2 - 16 > 0$

$f_{xx}(-1,-1) = 12 > 0$

$\Downarrow$

local min at $(-1,-1)$

Ex: Find + classify cp's of

$$f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Soln:

CPs

$$\begin{cases} f_x = 20xy - 10x - 4x^3 \overset{\text{set}}{=} 0 & \text{(i)} \\ f_y = 10x^2 - 8y - 8y^3 \overset{\text{set}}{=} 0 & \text{(ii)} \end{cases}$$

(i) $2x(10y - 5 - 2x^2) = 0$

$2x = 0$

$\boxed{x = 0}$

Plug into (ii)

$-8y - 8y^3 = 0$

$-8y(1 + y^2) = 0$

$-8y = 0$

$y = 0$

$(0,0)$ is a cp

$y^2 = -1$

NO real soln

$10y - 5 - 2x^2 = 0$

$\boxed{y = \frac{5 + 2x^2}{10} = \frac{1}{2} + \frac{x^2}{5}}$

Plug into (ii)

$10x^2 - 8\left(\frac{1}{2} + \frac{x^2}{5}\right) - 8\left(\frac{1}{2} + \frac{x^2}{5}\right)^3 = 0$

$\Downarrow$ calculator

$x \approx \pm 0.85, \pm 2.64$

$(0.85, 0.64)$ $(2.64, 1.89)$

$(-0.85, 0.64)$ $(-2.64, 1.89)$

$f_{xx} = 20y - 10 - 12x^2$ $\quad f_{yx} = 20x$

$f_{xy} = 20x$ $\quad f_{yy} = -8 - 24y^2$

$D = (20y - 10 - 12x^2)(-8 - 24y^2) - 20^2 x^2$

CPs

$(0,0)$: $D(0,0) = (-10)(-8) - 0 > 0$ $\Rightarrow$ local max at $(0,0)$

$f_{xx}(0,0) = -10 < 0$

$(0.85, 0.64)$: $D(0.85, 0.64) \approx -184 < 0 \Rightarrow$ saddle pt at $(0.85, 0.64)$

$(-0.85, 0.64)$: $D < 0 \Rightarrow$ saddle pt at $(-0.85, 0.64)$

$(2.64, 1.89)$: $D > 0$, $f_{xx}(2.64, 1.89) < 0 \Rightarrow$ local max at

$(-2.64, 1.89)$: $D > 0$, $f_{xx} < 0$ $\Rightarrow (\pm 2.64, 1.89)$