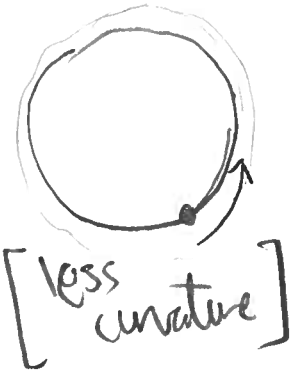
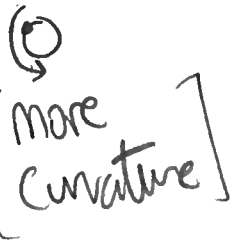
Curvature - measure of how curvy some curve is

κ k

$$k(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

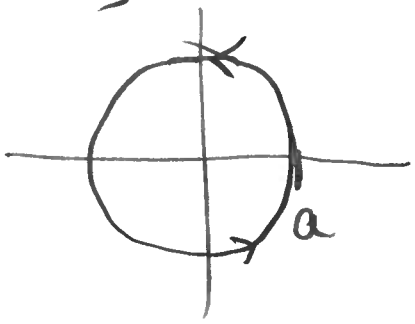
EX: Find curvature of circle of radius a.





small radius → big curv

big radius → small curv



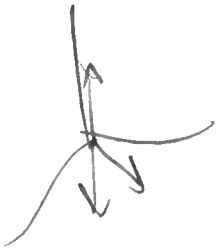
$$\begin{cases} \vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle \\ 0 \le t \le 2\pi \end{cases}$$

Use 1st formula

$$\vec{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$$

$$\vec{r}''(t) = \langle -a\cos(t), -a\sin(t), 0 \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin(t) & a\cos(t) & 0 \\ -a\cos(t) & -a\sin(t) & 0 \end{bmatrix}$$



$$= \langle 0, 0, a^2\sin^2(t) - (-a^2\cos^2(t)) \rangle$$

$$= \langle 0, 0, a^2 \rangle \qquad a^2(\underbrace{\cos^2 t + \sin^2 t}_{=1})$$

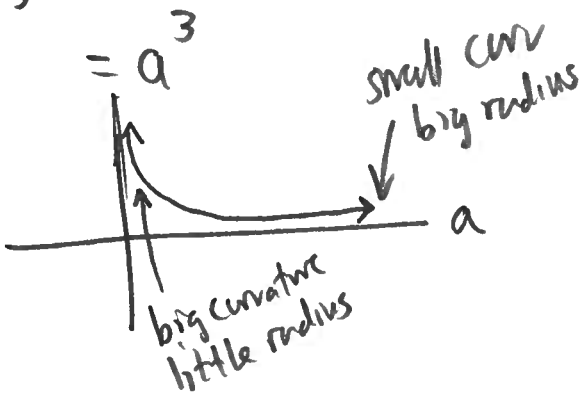
$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|\langle 0, 0, a^2 \rangle\|$$

$$= \sqrt{0^2 + 0^2 + (a^2)^2} = a^2$$

$$\|\vec{r}'(t)\|^3 = \|\langle -a\sin(t), a\cos(t), 0 \rangle\|^3$$

$$= \left(\underbrace{\sqrt{a^2}}_{=a}\right)^3 = a^3$$

$$k(t) = \frac{a^2}{a^3} = \frac{1}{a}$$



Ex: $\vec{r}(t) = \langle -2t, 8t, 1-7t^2 \rangle$

$\vec{r}'(t) = \langle -2, 8, -14t \rangle$

$\vec{r}''(t) = \langle 0, 0, -14 \rangle$

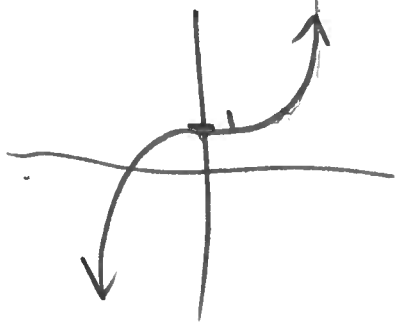
$\vec{r}' \times \vec{r}'' = \langle -14(8) - (-14t)(0), -[28-0], 0 \rangle$

$= \langle -112, -28, 0 \rangle$

$$\frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{\sqrt{(-112)^2 + (-28)^2}}{\left(\sqrt{4 + 64 + (-14t)^2}\right)^3}$$

$$= \frac{\sqrt{13328}}{\left(\sqrt{68 + 196t^2}\right)^3}$$

Ex: Find curvature of 2D plot $y = 3t^3 + 1$



We decide how to parametrize:

$$\vec{r}(t) = \langle t, 3t^3+1, 0 \rangle$$

$$\vec{r}'(t) = \langle 1, 6t^2, 0 \rangle$$

$$\vec{r}''(t) = \langle 0, 12t, 0 \rangle$$

$$\vec{r}' \times \vec{r}'' = \langle 0, -0, 12t - 0 \rangle = \langle 0, 0, 12t \rangle$$

$$\|\vec{r}' \times \vec{r}''\| = 12|t| \qquad \sqrt{144t^2}$$

$$\kappa(t) = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{12|t|}{\left(\sqrt{1+36t^4}\right)^3}$$

$$\|\vec{r}'\| = \sqrt{1^2 + (6t^2)^2} = \sqrt{1+36t^4}$$

Calculus

✓ $f: \mathbb{R} \to \mathbb{R}$ Calc 1-2

✓ $f: \mathbb{R} \to \mathbb{R}^n$ calc of space curves

→ $f: \mathbb{R}^n \to \mathbb{R}$ functions of several variables

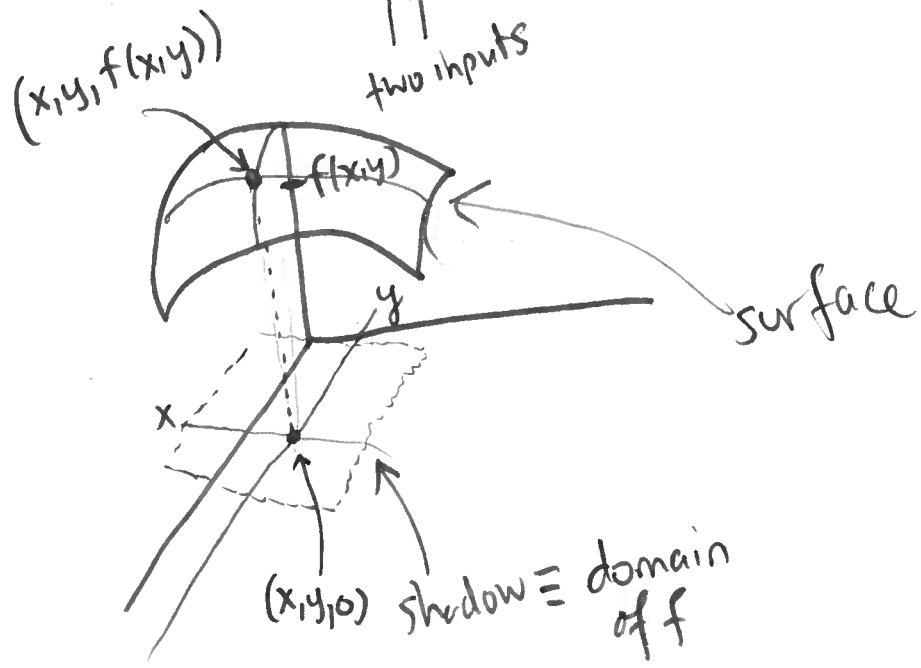
---

Functions of several variables

usually:

$$z = f(x,y)$$

two inputs



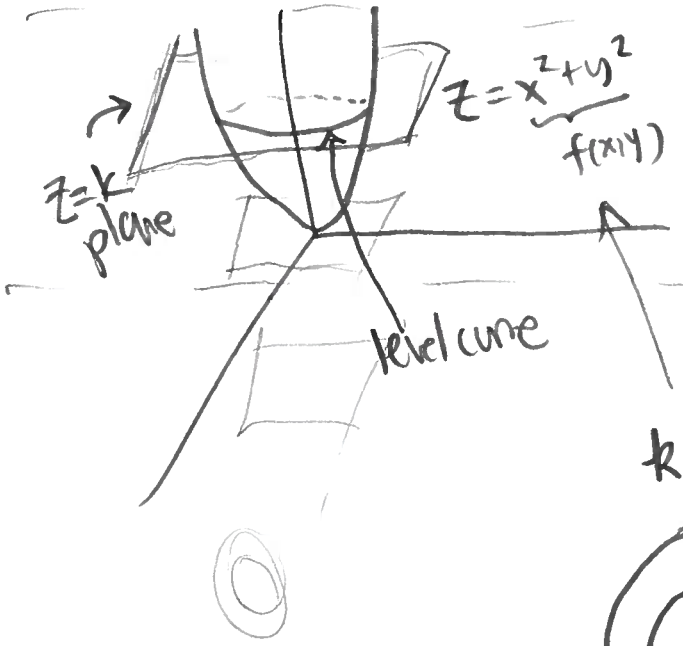
Ex: $f(x,t) = 18tx^4$

What is $f(2,1)$?

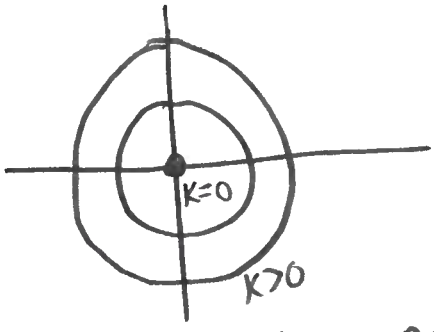
$f(2,1) = 18(1)\cdot 2^4 = 18\cdot 16 = 288$

($x=2$, $t=1$)

Level curves (just mean $z=k$ traces)



$k = x^2+y^2$

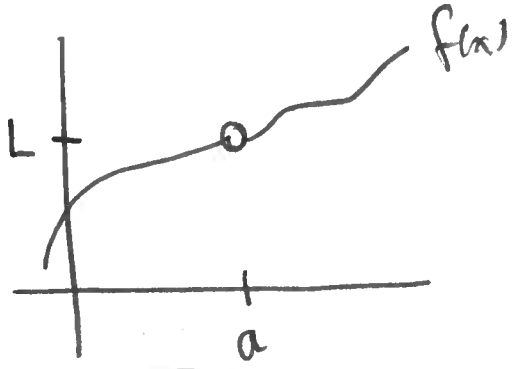


$k<0 \rightarrow$ empty — no level curve

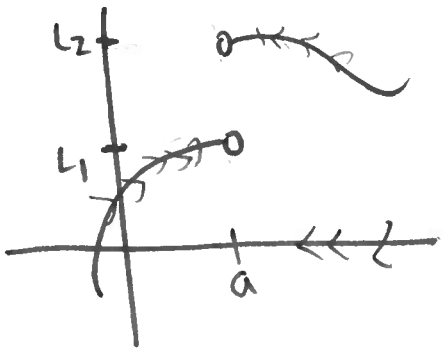
$-1 = x^2+y^2$

NO Real solutions

Recall Case 1:



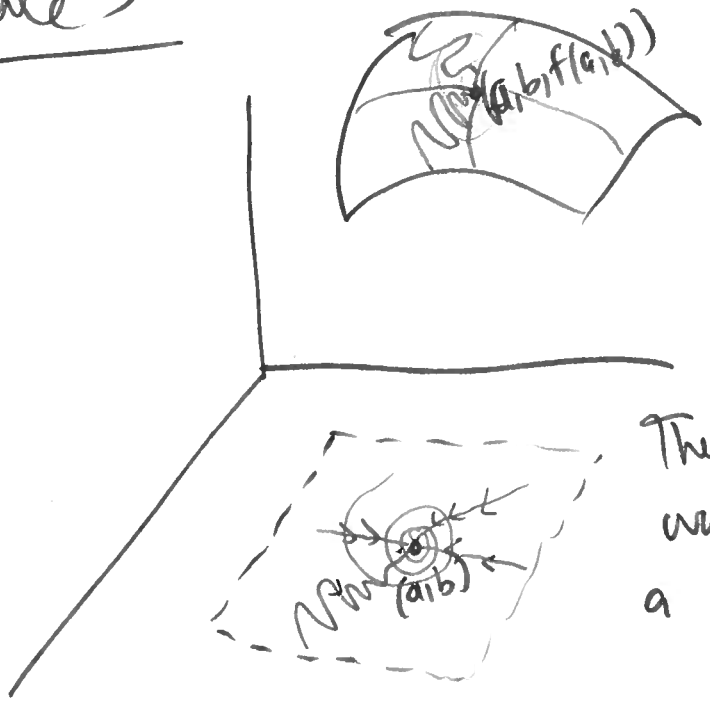
$$\lim_{x \to a} f(x) = L$$



$$\lim_{x \to a^-} f(x) = L_1$$

$$\lim_{x \to a^+} f(x) = L_2$$

Case 3



There are ∞-many ways to approach a point here!